Some Results of Differential Subordination and Superordination for Univalent Functions

Ihsan Ali Abbas
Ministry of Education, Education of Al-Qadisiyah, Diwaniyah, Iraq.

E-mail: ihasnabbas89@gmial.com.

Received 02-01-2021, Accepted 16-03-2021, published 14-04-2021.

DOI: 10.52113/2/08.01.2021/91-97

Abstract: Let q_1 and q_2 belong to a certain class of normalized analytic univalent functions in the open unit disk of the complex plane. Sufficient conditions are obtained for normalized analytic functions p to satisfy the double subordination chain $q_1(z) \prec p(z) \prec q_2(z)$, then we obtain $q_1(z)$ is the best subordinant, $q_2(z)$ is the best dominant. Also we derive some sandwich –type result.

Key words: Univalent function ,Differential subordination ,Differential superordination , Sandwich theorem

2021 Mathematics Subject Classification : 30 C 45.

1. Introduction

Let $\mathcal{H}$ be the class consisting of analytic function in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1,2,3,\dots\}$, $\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\}$ with $\mathcal{H}_1 = [1,1]$.
Le $\mathcal{X}$ t be the subclass of $\mathcal{H}$ consisting of functions the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad z \in U. \tag{1.1}$$

Which are analytic and univalent in U and satisfying the normalized conditions $f(0) = f'(z) - 1 = 0$. A function $f \in \mathcal{X}$ is said to be starlike function of order α, if $\mathcal{R}\{zf'(z)/f(z)\} > \alpha , 0 \le \alpha < 1 , z \in U$. Denote the class of starlike functions by $S^*(\alpha)$. In particularly the class $S^*(0) = S^*$, see [Duren]. A function $f \in \mathcal{X}$ is said to be convex function of order α, if $\mathcal{R}\{1 + zf''(z)/f'(z)\} > \alpha$, $0 \le \alpha < 1$, $z \in U$. Denote the class of convex functions by $C(\alpha)$. In particularly the class $C(0) = C$. see [Duren].

A function $f \in \mathcal{H}$ is said to be subordinate to an analytic function $g \in \mathcal{H}$ or g superordinates f, written as $f(z) \prec g(z)$, $z \in U$, if there exists a schwars function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, satisfying $f(z) = g(w(z))$, $z \in U$. If the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$, see [Miller].

An exposition on the widely used theory of differential subordination, developed in the main by Miller and Mocanu, with numerous applications to univalent functions can be found in their monograph [8].

Let $p, h \in \mathcal{H}$ and $\phi(r,s,t;z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If $h(z)$ be univalent in U and $p(z)$ satisfies the second-order subordination

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \qquad (1.2)$$

then $p(z)$ is called a solution of the differential subordination (1.2). A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.2) is said to be the best dominant of (1.2).

Miller and Mocanu [9] also introduced the dual concept of differential superordination .

Let $p, h \in \mathcal{H}$ and $\phi(r,s,t;z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \qquad (1.3)$$

then $p(z)$ is called a solution of the differential superordination (1.3). An analytic function $q(z)$ is a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfyin (1.3). A univalent subordinant $\tilde{q}(z)$ satisfying $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.3) is said to be the best subordinant.

Miller and Mocanu [9] obtained sufficient conditions on h, q and ϕ for which the following differential implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \implies q(z) \prec p(z).$$

Ali et al. [2] and [4] gave several applications of first – order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U. In [3], they obtained applied differential superordination to functions defined by means of linear operators. Recently, Shanmugam et al [10,11] obtained it called sandwich results for certain classes of analytic functions.

Hints to the work done by the authors [5] and [7] with formula special only on differential subordination. In [1] discusses around generalize this idea with formula special to (n) deriving only on differential subordination.

The following definition and results will be required in this search.

Definition 1.1.[9]: Let Q denote the set of all functions f that are analytic and univalent on $\overline{\mathrm{U}} \setminus E(f)$, where $E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$, and are such that $f'(z) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1.2.[8]: Let $q(z)$ be univalent in U and θ be analytic function in domain D containing $q(U)$. If $z\,q'(z)\,\theta(q(z))$ is starlike and

$$zp'(z)\,\theta\big(p(z)\big) \prec zq'(z)\,\theta\big(q(z)\big), \qquad (1.4)$$

then

$p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.4).

Lemma 1.3.[8]: Let $q(z)$ be univalent in U, and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, let $h(z) = \theta(q(z)) + Q(z)$. Suppose that (i) $Q(z)$ is a starlike function in U,

$$(ii)\ \Re\left\{\frac{z\,h'(z)}{Q(z)}\right\} > 0,$$

for all $z \in U$. If p(z) is analytic in U, with $p(0) = q(0)$, $p(U) \subseteq$ D and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \qquad (1.5)$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.5).

Lemma 1.4.[2]: Let $q(z)$ be univalent in U, and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(i) $Q(z) = zq'(z)\,\phi(q(z))$ is a starlike univalent in U,

$$(ii)\ \Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$$

, for all $z \in U$. If $p(z) \in \mathcal{H}[q(0),1]] \cap$ Q, with $p(U) \subseteq$ D, such that

$\theta(q(z)) + zq'(z)\phi(q(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \qquad (1.6)$$

then $q(z) \prec p(z)$, and $q(z)$ is the best subordinant of (1.6).

Lemma 1.5.[6]: Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$. Then $f \in S^*$ if and only if $zf'(z)/f(z) \in \mathbb{P}$ (where $\mathbb{P}$ is the class of all function φ analytic and having positive real part in D, with $\varphi(0) = 1$).

2. Main Results

Theorem 2.1. Let the function $q(z)$ be univalent in U, $q(z) \neq 0$ and $zq'(z)\theta(q(z)) \neq 0$ is starlike function in U. If $f \in \mathcal{X}$ satisfies the subordination

$$1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{zq'(z)}{\delta q(z)}, \qquad (2.1)$$

then

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \prec q(z), \quad z \in U, \ \delta \in \mathbb{C}^*,$$

and $q(z)$ is the best dominant.

Proof. Define the function

$$p(z) = \left[\frac{zf'(z)}{f(z)}\right]^{\delta}, z \in U, \ \delta \in \mathbb{C}^*, \qquad ($$

then

$$zp'(z) = \delta\left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left[1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right]. \qquad (2.3)$$

Setting $\theta(w) = \frac{1}{w\delta}$ it can easily observed that $\theta(w)$ is analytic function in $\mathbb{C}^*$, then, we have, $\theta(p(z)) = \frac{1}{\delta p(z)}$ and $\theta(q(z)) = \frac{1}{\delta q(z)}$.

From (2.3) and simple a computation shows that

$$zp'(z)\theta(p(z)) = 1 + \frac{z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}, \qquad (2.4)$$

together (2.1) and (2.4), we get

$$zp'(z)\theta(p(z)) \prec \frac{zq'(z)}{\delta q(z)} = z\, q'(z)\theta(q(z)).$$

Thus by applying Lemma (1.2), we obtain $p(z) \prec q(z)$, by using (2.2), we have the required result, and $q(z)$ is the best dominant of.

Taking $q(z) = \sin Az, -1 \leq A \leq 1$, since the function $\sin z$ is entire function and injective, in theorem (2.1), we obtain the following Corollary.

Corollary 2.2. If $f \in \mathcal{X}$ satisfies the subordination

$$1 + \frac{z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{Az}{\delta}\cot Az, \qquad (2.5)$$

then

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \prec \sin Az, \quad z \in U, \delta \in \mathbb{C}^*,$$

and $q(z) = \sin Az$ is the best dominant.

Theorem 2.3. Let $0 < \delta < 1$, and $\lambda, \beta \in \mathbb{C}^*$, let $q(z)$ be univalent in U and q satisfy the following condition :

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{\beta}{\lambda}\right\} > 0. \qquad (2.6)$$

If $f \in \mathcal{X}$ satisfies the subordination

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left\{\delta\lambda\left[1 + \frac{z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \beta\right\} \prec \beta q(z) - \lambda z q'(z), \qquad (2.7)$$

then

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \prec q(z), \qquad z \in U, \delta \in \mathbb{C}^*,$$

and $q(z)$ is the best dominant.

Proof. we begin by setting

$$p(z) = \left[\frac{zf'(z)}{f(z)}\right]^{\delta}, z \in U, \delta \in \mathbb{C}^*, \qquad ($$

then by a computation shows that

$$zp'(z) = \delta\left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left[1 + \frac{z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right]. \qquad (2.9)$$

Let us consider the function

$$\theta(w) = \beta w, \ \phi(w) = -\lambda, \quad w \in \mathbb{C},$$

then $\theta(w)$ and $\phi(w)$ is analytic in $\mathbb{C}$. Also if, we suppose

$$Q(z) = zq'(z)\phi(q(z)) = -\lambda z q'(z) \text{ and },$$

$h(z) = \theta(q(z)) + Q(z) = \beta q(z) - \lambda z q'(z)$.

From assumption (2.6), we yield that $Q(z)$ is starlike function in U, since

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)},$$

$$and \quad , \Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \Re\left\{\frac{\beta}{\lambda}\right\}$$

and by Lemma (1.5), with

$$\varphi(z) = \frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} \quad , \quad \varphi(0) = 1.$$

and we get

$$\Re\left\{\frac{z\,h'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{\beta}{\lambda}\right\} > 0 \ , \quad z \in U\,.$$

A simple computation together with (2.9) and (2.10), we have ,

$$zp'(z)\phi(p(z)) + \theta(p(z)) = \left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left\{\delta\lambda\left[1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \beta\right\},$$

therefore the subordination (2.7) becomes

$$zp'(z)\phi(p(z)) + \theta(p(z)) \prec zq'(z)\phi(q(z)) + \theta(q(z))\,.$$

By applying Lemma (1.3) and using (2.8) , we obtain our result.

Let us assume $q(z) = e^{\lambda Az}$ $-1 \leq A \leq 1$ in the Theorem (2.3), we have the following result.

Corollary 2.4. Let $f \in \mathcal{X}$ and suppose that $\Re\left\{1 + \lambda Az + \frac{\beta}{\lambda}\right\} > 0$ If f satisfies the subordination

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left\{\delta\lambda\left[1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \beta\right\} \prec (\beta + \lambda^2 zA)e^{\lambda Az}, \qquad (2.11)$$

then

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \prec e^{\lambda Az}, \qquad z \in U \ , \delta \in \mathbb{C}^*,$$

and $q(z) = e^{\lambda Az}$ is the best dominant.

Theorem 2.5. Let $q(z)$ be univalent in U with $q(0) = 1$, let $0 < \delta < 1$, and $\lambda, \beta \in \mathbb{C}^*$ and $\Re\{\theta'(q(z))/\phi(q(z))\} > 0$. Let $f \in \mathcal{X}$ and $\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \in \mathcal{H}[q(0), 1] \cap \mathrm{Q}$.

If the function

$$\Upsilon(z) = \left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left\{\delta\lambda\left[1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \beta\right\}, \qquad (2.12)$$

and

$$\beta q(z) - \lambda zq'(z) \prec \Upsilon(z) \qquad (2.13)$$

then

$$q(z) \prec \left[\frac{zf'(z)}{f(z)}\right]^{\delta}, \qquad z \in U \ , \delta \in \mathbb{C}^*,$$

and $q(z)$ is the best subordinant.

Proof. Consider the analytic function

$$p(z) = \left[\frac{zf'(z)}{f(z)}\right]^{\delta}, z \in U \ , \delta \in \mathbb{C}^*,$$

By differentiating (2.14), with respect to z, yields

$$zp'(z) = \delta\left[\frac{zf'(z)}{f(z)}\right]^{\delta}\left[1 + \frac{z\,f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right]. \qquad (2.15)$$

Setting the function.

$$\theta(w) = \beta w, \ \phi(w) = -\lambda, \quad w \in \mathbb{C},$$

then $\theta(w)$ and $\phi(w)$ is analytic in $\mathbb{C}$, with $\phi(w) \neq 0$ for all $w \in \mathbb{C}$. Also, we have

$Q(z) = zq'(z)\phi(q(z)) = -\lambda zq'(z)$, it is starlike univalent function in U, according to Lemma (1.5). And

$$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\beta q'(z)}{\lambda}\right\} > 0\,, z \in U\,,$$

by simple computation, shows that

$$\Upsilon(z) = \beta p(z) - \lambda zp'(z). \qquad (2.16)$$

From (2.13) and (2.16), with applying of Lemma (1.4), we obtain $q(z) \prec p(z)$, and using (2.14), we have required result.

Combining results of differential subordination Theorem 2.3 and superordination Theorem 2.5, to get at the following sandwich result.

Theorem 2.6. Let $q_1(z)$ and $q_2(z)$ be univalent functions in U, with $q_1(0) = q_2(0) = 1$, $\delta, \lambda \in \mathbb{C}^*$, $\beta \in \mathbb{C}$. Suppose q_1 satisfies $\Re\left\{\frac{\beta q'(z)}{\lambda}\right\} > 0$ and q_2 satisfies (2.6) . Let $f \in \mathcal{X}$ and

$$\left[\frac{zf'(z)}{f(z)}\right]^{\delta} \in \mathcal{H}[q(0), 1] \cap Q.$$

If the function $\Upsilon(z)$ given by equation (2.12) is univalent in U , and

$$\beta q_1(z) - \lambda zq_1'(z) \prec \Upsilon(z) \prec \beta q_2(z) - \lambda zq_2'(z),$$

then

$$q_1(z) \prec \left[\frac{zf'(z)}{f(z)}\right]^{\delta} \prec q_2(z) \qquad (2.17)$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (2.17).

References:

[1] W. G. Atshan and I. A. Abbas, **Differential Subordination for Univalent Functions**, European Journal of Scientific Research (EJSR) (UK), 145 (4) (2017), 427- 434.

[2] R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramanian, **Differential sandwich theorems for certain analytic functions** , Far East Journal of Mathematical Sciences, 15(1)(2004) ,87-94.

[3] R. M. Ali,V. Ravichandran, M. H. Khan and K.G. Subramanian, **Applications of first order differential superordinations to certain linear operators** , Southeast Asian Bulletin of Mathematics, 30(5)(2006) , 799-810.

[4] M. K . Aouf and T. Bulboacâ , **Subordination and superordination properties of multivalent function defined by certain integral operator**,J. Franklin Inst.347(2010), 641-653.

[5] W.G . Atshan and F.J . Abdulkhadim, **On subordination properties of univalent functions**, International Journal of Science and Research (IJSR), 3(9)(2014), 3 pages.

[6] P. L. Duren, **Univalent Functions**, in: Grundlehren der Mathematischen Wissenschaften , Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, (1983).

[7] M. H. Lafta, **On Some Classes of Topics in Univalent and Multivalent Function**

Theory, M. Sc. Thesis, University of Al- Qadisiyah, College of Computer Science &

Mathematics, Iraq, (2015).

[8] S. S. Miller and P. T. Mocanu, **Differential subordinations: Theory and Applications** ,

in : Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol.225, Marcel

Dekker, Incorporated, New York and Basel, (2000).

[9] S. S. Miller and P. T. Mocanu, **Subordinations of differential superordinations**,

Complex Variables, 48(10)(2003),815-826.

[10] T.N. Shanmugam, S. Sivasubbramanian and H. M. Srivastava, **On sandwich theorems for**

some classes of analytic functions, Int. J. Math. Math. Sci. Vol. (2006), Article ID 29684,

1-13.

[11] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, **Differential sandwich**

theorems for some subclasses of analytic functions, the Australian Journal of

Mathematical Analysis and Applications , 3(1)(2006) , 1-11.