Some Result of Fuzzy Separation Axiom in Fuzzy Topological ring Space

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Abstract: In this paper, we study fuzzy separation axiom $T_i,\ i = 0,1,2,3$ in the fuzzy topological ring space. Also the relationship between the types of fuzzy separation axiom was studied.

Keywords: Fuzzy topological ring; fuzzy T_i space, $i = 0,1,2,3$

1. Introduction:

In 1965 [11], Zadeh L. A. gave the definition of fuzziness. After three years C. Chang [2] gave the notion of fuzzy topology. In 1990[1], Ahsanullah and Ganguli, depended on the convergent in fuzzy topological space in the sense of Lowen[7, 8] to introduce the concept of fuzzy nbhd rings which gives the necessary and sufficient condition for a prefilter basis to be fuzzy nbhd prefilter of 0 in fuzzy topological ring. Also they are study the notions of right and left bounded fuzzy set and precompact fuzzy set fuzzy nbhd rings.

In 2009, Deb Ray, A. and Chettri, P [3] introduced fuzzy topology on a ring. Also in [4] they introduced fuzzy continuous function and studied left fuzzy topological ring. Our working to study fuzzy separation axiom $T_i,\ i = 0,1,2,3$ in the fuzzy topological ring space and obtaining the relationship between $T_i,\ i = 0,1,2,3$ spaces in the fuzzy topological ring space

For rich the paper, some basic concept of fuzzy set , fuzzy topology and fuzzy topological ring are given below. The symbol I will denote to the closed interval [0,1].

1.1 Definition [11]

A fuzzy set in R is a map $\partial: R \to I$ and, that is, belonging to I^R(the set of all fuzzy set of R) . Let $E \in I^R$, for every $r \in R$, we expressed by $E(r)$ of the degree of membership of r in R .

If $E(r)$ be an element of $\{0,1\}$, then E is said a crisp set.

1.2 Definition [2]

A class $\mu \in I^R$ of fuzzy set is called a fuzzy topology for R if the following are satisfied

1) $\emptyset, R \in \mu$
2) $\forall E, H \in \mu \rightarrow E \wedge H \in \mu$
3) $\forall (E_j)_{j\in J} \in \mu \rightarrow \vee_{j\in J} E_j \in \mu$

(R,μ) is called fuzzy topological space. if $A \in \mu$ Then A is fuzzy open and A^c (complement of A) is a fuzzy closed set.

1.3 Definition [1, 3]

A pair (R, μ) , where R a ring and μ be a fuzzy topology on R, is called fuzzy topological ring if the following maps are fuzzy continuous:

1) $R \times R \rightarrow R, (r, k) \rightarrow r + k.$
2) $R \rightarrow R,\ r \rightarrow -r$
3) $R \times R \rightarrow R, (r, k) \rightarrow r.k$

1.4 Definition [4]

A family B of fuzzy nbhds of r_α, for $0 < \alpha \le 1$, is called a fund. system of fuzzy nbhds of r_α iff for any fuzzy nbhd V of r_α, there is $U \in B$ such that $r_\alpha \le U \le V$

1.5 Definition [4]

Let R be a ring and μ a fuzzy topological on R. Let U and V are fuzzy sets in R. We define $U + V$, $-V$ and $U.V$ as follows

$$(U+V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$$

$$-V(k) = V(-k)$$

$$(U.V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$$

1.6.Theorem [4]

If R is a fuzzy topological ring then there is a fundamental system of fuzzy nbhds B of 0 $(0 < \alpha \le 1)$, such that the conditions:

(i) $\forall U \in B$, then $-U \in B$
(ii) $\forall U \in B$, then U is symmetric
(iii) $\forall U, V \in B$, then $U \wedge V \in B$
(iv) $\forall U \in B$, there is $V \in B$ such that $V + V \le U$
(v) $\forall U \in B$, there is $V \in B$ such that $V.V \le U$
(vi) $\forall r \in R, \forall U \in B$, there is $V \in B$ such that a $r.V \le U$ and $V.r \le U$.

1.7 Definition [7]

(R,μ) is fully stratified fuzzy topology on R if the fuzzy topology μ on R contain all constant fuzzy set

1.8 Definition [10]

A fuzzy topological space (R,μ) is said to be fuzzy T_0-topological space iff $\forall r_\alpha, k_\alpha \in R, r \ne k, \exists\, U \in \mu$ such that either $U(r) = 1$ and $U(k) = 0$ or $U(k) = 1$ and $U(r) = 0$.

1.10 Definition [10]

A fuzzy topological space $(R,\mu)$is said to be fuzzy T_1- topological space iff $\forall r_\alpha, k_\alpha \in R, r \ne k, \exists\, U, V \in \mu$ such that

$U(r) = 1$, $U(k) = 0$ and $V(r) = 0$, $V(k) = 1$

1.11 Definition [10]

A fuzzy topological space (R, μ) is said to be fuzzy Hausdorff or fuzzy T_2 space iff for any two distinct fuzzy points $r_\alpha, k_\alpha \in R$, there exists disjoint fuzzy sets $U, V \in \mu$ with

$$U(r) = V(k) = 1.$$

1.12 Definition [10]

A fuzzy topological space (R, μ) will be called fuzzy regular if for each fuzzy point r_α and each fuzzy closed set H such that $H(r) = 0$ there are fuzzy open sets U and V such that $U(r) > 0$, $H \leq V$and $U \wedge V = \emptyset$.

1.14 Proposition [10]

If a space R is a fuzzy regular space, then for any open set U and a fuzzy point $r_\alpha \in R$ such that $cl(U)(r) = 0$ there exists an open set V such that $\alpha \leq V \leq cl(V) \leq U$

1.15 Definition [10]

A fuzzy topological space(R, μ) will be called normal if for each pair of fuzzy closed sets H_1 and H_2 such that $H_1 \wedge H_2 = \emptyset$ there exist fuzzy open sets U_1 and U_2 such that $U_1 \leq H_1$ and $U_2 \leq H_2$ and $U_1 \wedge U_2 = \emptyset$.

1.16 Definition [10]

A fuzzy topological space (R, μ) is said to be fuzzy T_3-topological space iff it is fuzzy T_1 -and fuzzy regular.

2.Separation Axiom

2.1 Theorem

Let (R, μ) be a fuzzy topological ring. If $k_\alpha \in \{r_\alpha : R(r) = \max\{R(h)\}, \forall h \in R\}$ and U is a fuzzy nbhd of 0, then $k + U$ is a fuzzy nbhd of k such that $(k + U)(k) = \max\{R(h)\}$, $\forall h_\alpha \in R\}$.

Proof

Since U is a fuzzy nbhd of 0, there exists V fuzzy open set such that $V \subseteq U$ and $V(0) = U(0) = \max\{R(h)\}$, $\forall h \in R\}$. Let $g_k(r_\alpha): (R, \mu) \to (R, \mu)$, $g_k(r) = r_\alpha + k_\alpha$. g_k is a fuzzy homeo. Thus $k + V$ is a fuzzy open set.

$$k + V(k) = V(k - k) = V(0)$$
$$= \max\{R(h)\}, \quad \forall h \in R.$$
$$k + U(r) = U(r - k) \geq V(r - k) = k + V(r) \text{ for all } r \in R.$$

Thus there exists $k + V$ fuzzy open such that $k + V \leq k + U$ and $(k + V)(K) = (k + U)(k) = \max\{R(h)\}$, $\forall h \in R$

2.2 Theorem

Let (R, μ) be a fuzzy topological ring. If $k_\alpha \in \{r_\alpha : R(r) = \max\{R(h)\}, \forall h \in R\}$ and

U is a fuzzy nbhd of k_α such that $U(k) = \max\{R(h)\}$, $\forall h \in R$, then $U - k$ is a fuzzy nbhd of 0 such that $(U)(0) = \max\{R(h)\}$, $\forall h \in R$.

Proof

Since U is a fuzzy nbhd of k_α, there exists V fuzzy open set such that $V \subseteq U$ and $V(k) = U(k) = \max\{R(h)\}$, $\forall h \in R$. Let $g_k: R \to R$, $g_k(r_\alpha) = r_\alpha - k_\alpha$, g_k is a fuzzy homeo. Thus $-k + V$ is a fuzzy open set.

$V - k(0) = V(0 + k) = V(k) = \max\{R(h)\}$, $\forall h \in R\} = V(0)$.

$U - k(r) = U(r + k) \geq V(r + k) = V - k(r)$ for all $r \in R$.

Thus there exists $V - k$ fuzzy open set such that $V - k \leq U - k$ and $V(0) = U - k(0) = \max\{R(h)\}$, $\forall h \in R\}$.

2.3 Definition

A fuzzy topological ring (R, μ) is said to be Fuzzy T_0 – space iff for each $\mathrm{r}_\alpha, \mathrm{k}_\alpha$ s.t $\mathrm{r} \neq \mathrm{k}$ there exists fuzzy open set U s.t $\mathrm{r}_\alpha \in U$, $\mathrm{k}_\alpha \notin U$ or $\mathrm{k}_\alpha \in U$, $\mathrm{r}_\alpha \notin U$

2.4 Example

Let Z_2 be the ring of integers modulo 2. Define fuzzy sets E_1, E_2, E_3 on Z_2 as

$E_1([0]) = 0.9$, $E_1([1]) = 0$,

$E_2([0]) = 0$, $E_2([1]) = 0.9$

for all $r \in \mathrm{Z}_2$. Let $\mu = \{\emptyset, \mathrm{Z}_2, E_1, E_2\}$ is a fuzzy topological ring on Z_2, then (Z_2, μ) is a fuzzy T_0 – topological ring space.

2.5 Definition

A fuzzy topological ring (R, μ) is said to be Fuzzy T_1 – space iff for each $\mathrm{r}_\alpha, \mathrm{k}_\alpha$ s.t $\mathrm{r} \neq \mathrm{k}$ there exists fuzzy open sets U, V s.t $\mathrm{r}_\alpha \in U$, $\mathrm{k}_\alpha \notin U$ and $\mathrm{k}_\alpha \in V$, $\mathrm{r}_\alpha \notin V$

2.6 Example

Let Z_2 be the ring of integers modulo 2. Define fuzzy sets E_1, E_2, E_3 on Z_2 as

$E_1([0]) = 0.25$, $E_1([1]) = 0$,

$E_2([0]) = 0$, $E_2([1]) = 0.25$

$E_3([0]) = 0.25$, $E_3([1]) = 0.25$

for all $r \in \mathrm{Z}_2$. Let $\mu = \{\emptyset, \mathrm{Z}_2, E_1, E_2, E_3\}$ is a fuzzy topological ring on Z_2, then (Z_2, μ) is a fuzzy T_1 – topological ring space.

2.7 Definition

A fuzzy topological ring (R, μ) is said to be Fuzzy T_2 – topological ring space iff for any two fuzzy points $\mathrm{r}_\alpha, \mathrm{k}_\beta$ s.t $\mathrm{r} \notin \mathrm{supp}(\mathrm{k})$ and $k \notin supp(r)$, there exists two fuzzy open sets U, V s.t $\mathrm{r}_\alpha \in U$, $\mathrm{k}_\alpha \in V$ and $U \wedge V = \emptyset$

2.8 Example

Let Z_4 be the ring of integers modulo 4, with fuzzy discrete topology μ_D on it. The fuzzy topological ring (Z_4, μ_D) is Fuzzy T_2 – topological ring space.

2.9 Definition

A fuzzy topological ring (R, μ) is said to be Fuzzy regular space if $\forall r_\alpha \in R$ and a fuzzy closed set F with $F(r) = 0$ there exists fuzzy open sets U, V such that $r_\alpha \in U$ and $F \subset V$ and $\wedge V = \emptyset$.

2.10 Definition

A fuzzy topological ring (R, μ) is said to be Fuzzy T_3 −topological ring space if (R, μ) is fuzzy T_1 – space and fuzzy regular topological ring space.

2.11 Example

Let ℝ be the ring of real number, with fuzzy usual topology μ_U on it. Then $(\mathbb{R}, \mu_U)$ is Fuzzy T_3 − topological ring space

2.12 Theorem

If (R, μ) is fuzzy T_2- topological ring space then $\{0_\alpha\}$ is fuzzy closed subset in (R, μ).

Proof

Let (R, μ) is fuzzy T_2- topological ring space. For any $r_\alpha \neq 0_\alpha$ be another fuzzy point, assume $U(r) > 0$, $\forall U \in \{B_0\}$, then there exists fuzzy open set V of r_α s.t $V(0) = 0$, implies $\overline{\{0\}}(r) = 0$. Since r is arbitrary. Thus $\overline{\{0_\alpha\}} = \{0_\alpha\}$ and $\{0_\alpha\}$ is fuzzy closed set.

2.13 Theorem

For any fuzzy topological ring space (R, μ), if $\{0_\alpha\}$ is fuzzy closed subset in R and if B_0 is a basis of fuzzy nbhd of 0_α, then $\wedge_{V\in\{B_0\}} V = \{0_\alpha\}$.

proof

Let $\overline{\{0_\alpha\}}$ be an fuzzy closed set and $\{B_0\}$ be a basis of fuzzy nbhds of 0_α. Then by theorem 2.12, $\{0_\alpha\} = \overline{\{0_\alpha\}} = \wedge_{V\in\{B_0\}}(\{0_\alpha\} + V) = \wedge_{V\in\{B_0\}} V$ Thus $\wedge_{V\in\{B_0\}} V = \{0_\alpha\}$

2.14 Theorem

For any fuzzy topological ring space (R, μ), and B_0 is a basis of fuzzy nbhd of 0_α, if $\wedge_{V\in\{B_0\}} V = \{0_\alpha\}$, then (R, μ) is fuzzy T_0-topological ring space.

Proof

Let $\{B_0\}$ be a basis of fuzzy nbhds of 0_α and $\wedge_{V\in\{B_0\}} V = \{0_\alpha\}$. Let r_α and k_α are fuzzy points with different support. Therefor $r_\alpha - k_\alpha \neq 0$, so there exists $V \in \{B_0\}$, s.t $V(r-k) = 0$. Now by theorem 1.6, $k + V(r)$ is fuzzy nbhd of k_α and $(k+V)(r) = V(r-k) = 0$. Thus $r_\alpha \notin k + V$ and (R, μ) is fuzzy T_0-topological ring space.

2.15 Theorem

If (R, μ) is fuzzy T_0- topological ring space, Then (R, μ) is fuzzy T_1-topological ring space

Proof

Let (R, μ) be a fuzzy T_0-topological ring space and let r_α and k_α are fuzzy points with different support. Then there exists fuzzy open set V of 0_α s.t $(r+V)(k) = 0$ or $(k+V)(r) = 0$. We can assume that V is symmetric fuzzy open set of 0_α. To explain that $(r+V)(k) = 0$ if $(k+$

$V)(r) = 0$ we assume the contrary, suppose that $(r + V)(k) > 0$, therefor $(r - V)(k) > 0$. Implies

$$(r + V)(k) = V(k - r) = V\big(-(r - k)\big) = V(r - K) = (k + V)(r) > 0.$$

This contradiction, similarly if $(k + V)(r) = 0$. Thus (R, μ) is fuzzy T_1-topological ring space.

2.16 Theorem

If (R, μ) is fuzzy T_1-topological ring space then (R, μ) is fuzzy T_3-topological ring space.

Proof

Let (R, μ) is fuzzy T_1-topological ring space and let $r \in R$, F be a fuzzy closed subset in R s.t $F(r) = 0$, then $F^c(r) = 1$ and F^c is fuzzy open set. Therefor $F^c - r$ is an fuzzy open nbhd system of 0_α. Then there exists fuzzy open set V_0 of 0_α s.t $V_0 \le F^c - r$. Now

$$\overline{V_0} = \bigcap(V_0 + (F^c - r)) = \bigcap(F^c - r) = \{0_\alpha\},$$

$$(r + \overline{V_0})(k) = \overline{V_0}(k - r) = \min\{\overline{V_0}(k), \overline{V_0}(-r)\} = \min\{\overline{V_0}(k), \overline{V_0}(r)\} = \overline{V_0}(k) = V_0(k) \qquad , \forall k \in R$$

implies $r + \overline{V_0} \le F^c - r$. Thus (R, μ) is fuzzy regular and consequently it is fuzzy T_3-topological ring space.

2.17.Theorem

For any fuzzy topological ring (R, μ), the following conditions are equivalent

1) (R, μ) is fuzzy T_2-topological ring space
2) $\{0_\alpha\}$ is FZ.closed subset in R.
3) If B_0 is a basis of nbhd of 0_α, then $\bigcap_{V \in B_0} V = \{0_\alpha\}$
4) (R, μ) is fuzzy T_0-topological ring space.
5) (R, μ) is fuzzy T_1-topological ring space.
6) (R, μ) is fuzzy T_3-topological ring space.

Proof

By theorems 2.10, 2.11, 2.12, 2.13 and 2.14

2.18 Theorem

Let (R, μ) be a fuzzy topological ring, then Every fuzzy subspace of fuzzy T_0-space is a fuzzy $T_0 - space$.

Proof:

Obvious

2.19 Theorem

Let (R, μ) be a fuzzy topological ring and E is a fuzzy subring of (R, μ), if (E, μ_E) is fuzzy Hausdorff and $(R/E\,, \beta)$ is fuzzy Hausdorff then (R, μ) is fuzzy Hausdorff.

Proof

If R/E is fuzzy Hausdorff, then E is fuzzy closed set in (R, μ).

If also (E, μ_E) is fuzzy Hausdorff then $\{0_\alpha\}$ is fuzzy closed in E.

Hence $\{0_\alpha\}$ is fuzzy closed in (R, μ), and (R, μ) is fuzzy Hausdorff.

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