ESSENTIAL PROBLEMS FOR SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED ON UNIT DISK

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INTRODUCTION

Let \( \mathcal{H} \) denote the class of functions of the form

\[
f(w) = \sum_{k=2}^{\infty} a_k w^k , \quad a_k \geq 0 \quad (1)
\]

which are analytic in the open unit disk \( U = \{ w; \ w \in \mathbb{C} : |w| < 1 \} \). For the functions \( f \) and \( g \) in \( \mathcal{H} \), we say that \( f \) is subordinate to \( g \) in \( U \), and write \( f < g \) if there exists a function \( k(w) \) in \( U \) such that \( |k(w)| < 1 \) and \( k(0) = 0 \) with \( f(w) = g(k(w)) \) in \( U \). If \( f \) is univalent in \( U \), then \( f \sim g \) is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \), see[5].

Let \( f, g \in \mathcal{H} \) be given by

\[
g(w) = \sum_{k=2}^{\infty} b_k w^k b_k \geq 0 \quad \text{for all} \quad z \in U,
\]

then the Hadamard product (or convolution)

\[
(f * g)(w) = w + \sum_{k=2}^{\infty} a_k b_k w^k.
\]

For \( m \in \mathbb{N}, \lambda, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq b \leq 1, f \in \mathcal{H} \), Frasin operator [3].

\[
D_{m,b}^{\lambda} f(z) : \mathcal{H} \to \mathcal{H} \text{ is defined as follows}
\]

\[
D_{m,b}^{\lambda} f(w) = w + \sum_{k=2}^{\infty} \left[ 1 + (k - 1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} b^j \right]^\lambda a_k w^k
\]

One can easily proved that

\[
C_j^{m}(b) \left( D_{m,b}^{\lambda} f(w) \right) = D_{m+1,b}^{\lambda+1}(a, c) f(w) - (1 - C_j^{m}(b)) D_{m,b}^{\lambda} f(w) \quad (3)
\]

where

\[
C_j^{m}(b) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} b^j.
\]

To get our outcomes, we must recall a necessary known concepts.

**Definition (1) [4]:** Let \( \Omega \) and \( \Delta \) be any sets in \( \mathbb{C} \), let \( \delta \) be an analytic function in the open unit disk \( U \) with \( \delta(0) = a \) and let

\[
\psi(r, s, t; w) : \mathbb{C}^2 \times U \to \mathbb{C}.
\]

The core of this monograph manages speculation of the accompanying ramifications:

\[
\{ \psi(\delta(w), w\delta'(w), w^2\delta''(w)); w) : w \in U \} \subset \Omega \Rightarrow \delta(U) \subset \Delta. \quad (4)
\]

If \( \Delta \) is a simply connected domain containing the point \( a \) and \( \Delta \neq \mathbb{C} \), then there is a conformal mapping \( \sigma \) of \( U \) onto \( \Delta \) such that \( \sigma(0) = a \). In this case, (4) can be written as:

\[
\{ \psi(\delta(w), z\delta'(w), w^2\delta''(w)); w) : w \in U \} \subset \Omega \Rightarrow \delta(U) \subset \sigma(U).
\]
If $\Omega$ is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping $h$ of $U$ onto $\Omega$ such that $h(0) = \psi(a,0,0;0)$. Also, if the function $\psi(\delta(w),w\delta'(w),w^2\delta''(w);w)$ is analytic in $U$, then it can be written as:

$$\psi(\delta(w),w\delta'(w),w^2\delta''(w);w) < h(z) \Rightarrow \delta(z) < \sigma(z). \quad (5)$$

**Definition (2) [4]:** Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be univalent in $U$. If $\delta$ is analytic in $U$ and satisfies the second-order differential subordination

$$\psi(\delta(w),w\delta'(w),w^2\delta''(w);w) < h(w), \quad (6)$$

then $\delta$ is called a solution of the differential subordination. The univalent function $\sigma$ is called a dominant of the solutions of the differential subordination, relation (6) simply dominant if $\delta < \sigma$ for all $\delta$ satisfying (6).

A dominant $\tilde{\sigma}$ is said to be best dominant if it satisfies $\tilde{\sigma} < \sigma$ for all dominants $\sigma$ of (6).

**Definition (3) [4]:** Denote by $Q$ the set of all functions $\sigma$ that are injective and analytic on $U \setminus E(\sigma)$, where

$$E(\sigma) = \{ \xi \in \partial U : \lim_{z \to \xi} \sigma(z) = \infty \}, \quad (7)$$

and are such that $\sigma'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(\sigma)$. Further, let the subclass of $Q$ for which $\sigma(0) = a$ be denoted by $Q(a)$. $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$, where $Q_1 = \{ q \in Q : q(0) = 1 \}$.

**Definition (4) [4]:** Let $\Omega$ be a set in $\mathbb{C}$ and $\sigma \in Q$ and $n$ be positive integer. The class of admissible functions $\Psi_n[\Omega,\sigma]$ consist of those functions

$$\psi: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$$

which satisfy the admissibility condition

$$\psi(r,s,t;w,\xi) \notin \Omega. \quad (11)$$

Whenever $r = \sigma(\xi), s = k\xi \sigma'(\xi)$ and

$$\text{Re} \left\{ \frac{t}{s} + 1 \right\} \geq k \text{Re} \left\{ \frac{\xi \sigma''(\xi)}{\sigma'(\xi)} + 1 \right\}, \quad (8)$$

$w \in U, \xi \in \partial U \setminus E(\sigma), \xi \in \overline{U}$, and $k \geq n$. In particular case $\Psi_1[\Omega,\sigma] = \Psi[\Omega,\sigma]$.

**Definition (5) [4]:** Let $\Omega$ be a set in $\mathbb{C}$, $\sigma \in \mu[a,n]$ and $n$ be positive integer. The class of admissible function $\Psi_n[\Omega,\sigma]$ consist of those functions

$$\psi: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$$

that satisfies the admissibility condition

$$\psi(r,s,t;\xi,I) \notin \Omega.$$

Whenever $r = \sigma(w), s = (\frac{1}{m}) w \xi \sigma'(w)$ and

$$\text{Re} \left\{ \frac{t}{s} + 1 \right\} \geq \left( \frac{1}{m} \right) \text{Re} \left\{ \frac{w \xi \sigma''(w)}{\xi \sigma'(w)} + 1 \right\}, \quad (9)$$

$w \in U, \xi \in \partial U, \xi \in \overline{U}$, and $m \geq n \geq 1$. In particular case for $n=1$

$$\Psi'_1[\Omega,\sigma] = \Psi'[\Omega,\sigma] \quad (10)$$

Similar study is carried out by several authors, like Billing [1] Dihnggong and Liu [2], Oros [5,6], and Lupas [7,8]. Many authors studied different classes with essential problems such as Sarah A. AL-Ameedee, Waggas Galib Atshan & Faez Ali AL-Maamori [9], Serkan Çakmak, Sibel Yalcin, Şahsene Altinkaya [10], Waggas Galib Atshan and, Haneen Zaghir [11], Odeh Z. and Kassim A. Jassim [12], Odeh Z. and Kasim A. Jassim [13] and Odeh Z. and Kassim A. Jassim [14].

**Definition (6):** Let $\Omega$ be a set in $\mathbb{C}$ and $\sigma \in Q_0 \cap \mu[0,\delta]$. The class of admissible functions $\Phi_k[\Omega,\sigma]$ consists of those functions $\Phi: \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\Phi(u,v,w;\xi) \notin \Omega, \quad (11)$$

whenever

$$u = \sigma(\xi), v = k \xi \sigma'(\xi)$$

$$+ \left( 1 - C_j^m(T) \right) \sigma(\xi), (\lambda > -1),$$

and
\[ \begin{aligned}
&\text{Re} \left\{ w - (c_j^m(b)^2 - 2c_j^m(b) + 1)u - \left( 2 - c_j^m(b) \right) c_j^m(b)v \\
&\quad + c_j^m(b)^2 \left( \frac{v - (1 - c_j^m(b)) u}{c_j^m(b)} \right) \right\} \\
&+ 1 \geq kRe \left\{ \frac{\xi}{\sigma} + 1 \right\} \tag{12}
\end{aligned} \]

Then \( D_{m,b}^\lambda f(w) < \sigma(w). \)

**Proof:** By using (2) and (3), we get the equivalent relation
\[ D_{m,b}^{\lambda+1} f(w) = C_j^m(b)w D_{m,b}^\lambda f(w) + (1 - C_j^m(b)) D_{m,b}^\lambda f(w). \tag{14} \]

Assume that
\[ F(w) = D_{m,b}^\lambda f(w). \tag{15} \]

Then
\[ D_{m,b}^{\lambda+1} f(w) = C_j^m(b)w F'(w) + (1 - C_j^m(b)) F(w). \tag{16} \]

Therefore,
\[ D_{m,b}^{\lambda+2} f(w) = C_j^m(b)w \left( D_{m,b}^{\lambda+1} f(w) \right)' + (1 - C_j^m(b)) D_{m,b}^{\lambda+1} f(w). \tag{17} \]

Then we have by (13),
\[ \left( D_{m,b}^{\lambda+1} f(w) \right)' = C_j^m(b)w F''(w) \]
\[ + C_j^m(b) F'(w) + (1 - C_j^m(b)) F'(w). \tag{18} \]

So,
\[ D_{m,b}^{\lambda+2} f(w) = C_j^m(b)^2 w^2 F''(w) + \left( 2 - C_j^m(b) \right) WC_j^m(b) F'(w) + \left( C_j^m(b)^2 - 2C_j^m(b) + 1 \right) F(w). \tag{19} \]

Let \( u = r, v = C_j^m(b) s + \left( 1 - C_j^m(b) \right) r \)
\[ w = C_j^m(b)^2 t + \left( 2 - C_j^m(b) \right) C_j^m(b) s \]
\[ \quad + \left( C_j^m(b)^2 - 2C_j^m(b) + 1 \right) r \]

Assume that
\[ \psi(r, s, t; w) = \emptyset(u, v, w; \omega, \xi) \]
\[ = \emptyset \left( r, C_j^m(b)s \right) + (1 - C_j^m(b)) r \]
\[ + \left( 2 - C_j^m(b) \right) C_j^m(b)s + \left( C_j^m(b)^2 - 2C_j^m(b) + 1 \right) r; \]

By using (15) and (19), we obtain
\[ \psi(F(w), wF'(w), w^2 F''(w); w) \]
\[ = \emptyset \left( D_{m,T}^\lambda f(w), D_{m,T}^{\lambda+1} f(w), D_{m,T}^{\lambda+2} f(w); w \right). \tag{20} \]

Therefore, by making use (14), we get
\[ \psi(F(w), wF'(w), w^2 F''(w); w) \in \Omega. \tag{21} \]

Also, by using
\[ w = C_j^m(b)^2 t + \left( 2 - C_j^m(b) \right) C_j^m(b)s + \left( C_j^m(b)^2 - 2C_j^m(b) + 1 \right) r \]
\[ + 1 = \frac{t}{s} + 1 \tag{22} \]

and the admissibility condition for \( \emptyset \in \Phi_k[\Omega, \sigma] \) is equivalent to the admissibility condition for \( \psi \) then \( \psi \in \Psi_n \{ \Omega, \sigma \} \) and therefore \( F(w) < \sigma(w). \)

Hence, we get \( D_{m,T}^\lambda f(w) < \sigma(w). \)

If we assume that \( \Omega \neq C \) is a simply connected domain. So, \( \Omega = h(U) \), for some conformal mapping \( h \) of \( U \) onto \( \Omega \). Assume the class is written as \( \Phi_k[h, \sigma]. \) Therefore, we conclude immediately the following theorem.

**Theorem (2):** Let \( \emptyset \in \Phi_k[h, \sigma]. \) If \( f \in H \) satisfies
\[ \emptyset(D_{m,T}^\lambda f(w), D_{m,T}^{\lambda+1} f(w), D_{m,T}^{\lambda+2} f(w); w) < h(w), \tag{23} \]
then \( D_{m,T}^\lambda f(w) < \sigma(w). \)

The next result is an extension of Theorem (1) to the case where the behavior of \( \sigma \) on \( \partial U \) is unknown.

**Corollary (1):** Let \( \Omega \subset C, \sigma \) be univalent in \( U \) and \( \sigma(0)=0. \) Let \( \emptyset \in \Phi_k[\Omega, \sigma], \) for some \( \rho \in (0, 1), \) where \( \sigma_p(w) = \sigma(\rho w). \) If \( f \in H \) satisfies
If $f \in \mathcal{H}$ satisfies (23), then $D_{m,b}^\lambda f(w) < q(w)$ and $\sigma$ is the best dominant.

**Proof:** By using Theorem (2) and Theorem (3), we get that $\sigma$ is a dominant of (23). Since $\sigma$ satisfies (25), it is also a solution of (23) and therefore $\sigma$ will be dominant by all dominants of (23). Hence, $\sigma$ is the best dominant of (23).

**Definition (7):** Let $\Omega$ be a set in $\mathbb{C}$ and $\sigma \in \mathbb{Q}_0 \cap \mu_0$. The class of admissible functions $\Phi_{k,1}[\Omega, \sigma]$ consists of those functions $\Theta: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:
\[
\Theta(u, v, w; w) \in \Omega,
\]
whenever
\[
u = \sigma(\zeta), \quad v = kC_j^m(b)\sigma'(\zeta)
\]
\[
+ \sigma(z), (\lambda > -1),
\]
and
\[
\Re \left\{ \left( \frac{w-u}{vC_j^m(b)} \right)^2 - \frac{(C_j^m(b) + 2)}{C_j^m(b)} + 1 \right\} \geq kRe \left\{ \frac{\sigma''(\zeta)}{\sigma'(\zeta)^2} \right\} + 1 \tag{26}
\]

**Theorem (5):** Let $\Theta \in \Phi_{k,1}[\Omega, \sigma]$. If $f \in \mathcal{H}$ satisfies
\[
\left\{ \Theta \begin{pmatrix} D_{m,b}^\lambda f(w) \\ w \\ D_{m,b}^{\lambda+1} f(w) \\ w \\ D_{m,b}^{\lambda+2} f(w) \\ w \\ w \end{pmatrix} \right\} \subset \Omega,
\]
then
\[
\frac{D_{m,b}^\lambda f(w)}{w} < \sigma(w).
\]

Proof: Let the analytic function $\Theta$ in $U$ be defined by
\[
\Theta(w) = \frac{D_{m,b}^\lambda f(w)}{w} \tag{28}.
\]
Then by (13) and (28), we get
\[
D_{m,b}^{\lambda+1} f(w) = C_j^m(b)w \left( D_{m,b}^\lambda f(w) \right)' + (1 - C_j^m(b))D_{m,b}^\lambda f(w)
\]
\[
= C_j^m(b)wF'(w) + F(w) + (1 - C_j^m(b))wF(w)
\]
\[
= C_j^m(b)wF'(w) + F(w) + (1 - C_j^m(b))wF(w)
\]
\[
D_{m,b}^{\lambda +2}f(w)w = C_j^m(b)2w^2F''(w) + F'(w)2wC_j^m(b) + wF'(w)C_j^m(b) + F(w)C_j^m(b)
+ (C_j^m(b)w2F'(w) + wF(w) - C_j^m(b)^2)bF'(w) + F(w) - C_j^m(b)^2(b)wF'(w)
- F(w)C_j^m(b)
\]

So, let define the transformation from \( C^3 \) to \( C \) by
\[ w = C_j^m(b)s + r \]

\[ v = C_j^m(b)s + r \]

Then
\[
\frac{D_{m,b}^{\lambda +2}f(w)}{w} = C_j^m(b)2w^2F''(w) + F'(w)2wC_j^m(b) + wF'(w)C_j^m(b) + F(w)C_j^m(b)
+ (C_j^m(b)w2F'(w) + wF(w) - C_j^m(b)^2)bF'(w) + F(w) - C_j^m(b)^2(b)wF'(w)
- F(w)C_j^m(b)
\]

\[ D_{m,b}^{\lambda +2}f(w)w = C_j^m(b)2w^2F''(w) + F'(w)2wC_j^m(b) + wF'(w)C_j^m(b) + F(w)C_j^m(b)
+ (C_j^m(b)w2F'(w) + wF(w) - C_j^m(b)^2)bF'(w) + F(w) - C_j^m(b)^2(b)wF'(w)
- F(w)C_j^m(b)
\]

This implies that
\[
D_{m,b}^{\lambda +2}f(w)w = C_j^m(b)(wF'(w) + F(w))
+ (1 - C_j^m(b))F(w)
= C_j^m(b)w(F'(w)C_j^m(b) + wF(w)) + (1 - C_j^m(b))C_j^m(b)w2F'(w) + C_j^m(b)wF(w) + wF(w)
- C_j^m(b)wF(w))
= C_j^m(b)w(C_j^m(b)w2F'(w) + wF(w))' + (1 - C_j^m(b))(C_j^m(b)w2F'(w) + wF(w))
= C_j^m(b)w(C_j^m(b)w2F'(w) + wF(w))' + (1 - C_j^m(b))(C_j^m(b)w2F'(w) + wF(w))
= w(C_j^m(b)2w^2F''(w) + F'(w)2wC_j^m(b) + wF'(w)C_j^m(b)
+ F(w)C_j^m(b)
+ ((C_j^m(b)w2F'(w) + wF(w) - C_j^m(b)^2)bF'(w) + F(w) - C_j^m(b)^2(b)wF'(w)
- wF(w)C_j^m(b))
\]

Then
h(\rho \omega). Let \( \emptyset: \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) satisfy one of the following conditions:

1. \( \emptyset \in \Phi_k[\Omega, \sigma] \) for some \( \rho \in (0,1) \) or
2. there exists \( \rho_0 \in (0,1) \) such that \( \emptyset \in \Phi_k[h_\rho, \sigma] \) for all \( \rho \in (\rho_0, 1) \).

If \( f \in \mathcal{H} \) satisfies (23), then
\[
\frac{D_{m,b}^\lambda f(w)}{w} < \sigma(w).
\]

**Proof:** **Case (1):** By using Theorem (1), we get
\[
D_{m,T}^\lambda f(w) < \sigma. \text{Since } \sigma(w) < q(w) \text{ then we get the result.}
\]

**Case (2):** Assume that \( \mathcal{F}(w) = D_{m,T}^\lambda f(w) \) and \( \mathcal{F}_p(w) = \mathcal{F}(\rho w) \). So,
\[
\emptyset(\mathcal{F}(\rho w), w\mathcal{F}'(w), w^2\mathcal{F}''(w); w \rho w) = \emptyset(\mathcal{F}(\rho w), \rho w\mathcal{F}'(w), \rho^2 w^2\mathcal{F}''(w); w \rho w) \in h_p(U).
\]
By using Theorem (1) and associated with
\[
\emptyset(\mathcal{F}(w), w\mathcal{F}'(w), w^2\mathcal{F}''(w); w(w)) \in \Omega, \text{where } w \text{ is any function mapping from } U \text{ onto } U, \text{ with } w(w) = \rho w, \text{ we obtain } \mathcal{F}_p(w) < \sigma_p(w) \text{ for } \rho \in (\rho_0, 1). \text{ By letting } \rho \rightarrow 1^-, \text{ we get }
\[
\frac{D_{m,b}^\lambda f(w)}{w} < \sigma(w).
\]
The next theorem gives the best dominant of the differential subordination (23).

**Theorem (8):** Let \( h \) be univalent in \( U \) and let \( \emptyset: \mathbb{C}^3 \times U \rightarrow \mathbb{C} \). Suppose that the differential equation
\[
\emptyset(\sigma(w), \sigma'(w), w^2 \sigma''(w); w) = h(w)
\]
has a solution \( \sigma \) with \( \sigma(0) = 0 \) and satisfy one of the following conditions:

1. \( \sigma \in Q_0 \) and \( \emptyset \in \Phi_k[h, \sigma] \).
2. \( \sigma \) is univalent in \( U \) and \( \emptyset \in \Phi_k[h, \sigma] \) for some \( \rho \in (0,1) \).
3. \( \sigma \) is univalent in \( U \) and there exists \( \rho_0 \in (0,1) \) such that \( \emptyset \in \Phi_k[h_\rho, \sigma] \) for all \( \rho \in (\rho_0, 1) \). If \( f \in \mathcal{H} \) satisfies (23), then
\[
\frac{D_{m,b}^\lambda f(w)}{w} < \sigma(w) \text{ and } \sigma \text{ is the best dominant.}
\]

**REFERENCE**

12. Odeh Z. and Kassim A. Jassim, On the class of multivalent analytic functions defined by differential operator for derivative of first
