ESSENTIAL PROBLEMS FOR SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED ON UNIT DISK

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Abstract: In this paper, we investigate subordinate problems for subclass related to known functions convoluted with Frasin operator. Therefore, major results and special cases are obtained.

Keyword: analytic function , Hadamard product, univalent function , Differential subordination.

INTRODUCTION

Let $\mathcal{H}$ denoted the class of functions of the form

$$f(w)=\sum_{k=2}^{\infty} a_k\, w^k \quad , \ a_k \geq \mathbf{0} \qquad (1)$$

which are analytic in the open unit disk $U=\{w;\ w\ \in\ \mathbb{C} : |w|<\ 1\}$.For the functions f and g in $\mathcal{H}$, we say that f is subordinate to g in U, and write $f \prec g$ if there exists a function $k(w)$ in U such that $|k(w)|<1$ and $k(0)\ =\ 0$ with $f(w)=g(k(w))$ in U. If f is univalent in U, then $f \prec g$ is equivalent to $f(0)=g(0)$ and $f(U)\ \subset\ g(U)$, see[5]

Let $f, g \in \mathcal{H}$ be given by

$g(w\,)=\sum_{k=2}^{\infty} b_k\, w^k\ b_k \geq\ \mathbf{0}$, for all $z \in U$, then the Hadamard product (or convolution)

$$(\,f * g)(w)=w+\sum_{k=2}^{\infty} a_k\, b_k w^k.$$

For $m\ \in\ \mathbb{N}$, λ ,j$\in \mathbb{N}_0=\mathbb{N}\cup\{0\}$, $0 \leq b \leq 1$, $f \in \mathcal{H}$, Frasin operator [3] . $D_{m,b}^{\lambda} f(z)\!: \mathcal{H} \rightarrow \mathcal{H}$ is defined as follows

$$D_{m,b}^{\lambda} f(w)=w+\sum_{k=2}^{\infty}[1+(k-1)\sum_{j=1}^{m}\binom{m}{j}(-1)^{j+1}b^j]^{\lambda}\, a_k w^k \qquad (2)$$

One can esily proved that

$$C_j^m(b)\left(D_{m,b}^{\lambda} f(w\,)\right)' = D_{m,b}^{\lambda+1}(a,c) f(w\,) - (1-C_j^m(b)) D_{m,b}^{\lambda} f(w) \qquad (3)$$

where

$$C_j^m(b)=\sum_{j=1}^{m}\binom{m}{j}(-1)^{j+1}b^j.$$

To get our outcomes, we must recall a necessary known concepts.

<u>Definition(1) [4]:</u> Let Ω and Δ be any sets in $\mathbb{C}$, let δ be an analytic function in the open unit disk U with $\delta(0)$=a and let $\psi(r,s,t;w)\!: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$.

The core of this monograph manages speculation of the accompanying ramifications :

$$\{\psi(\delta(w), w\delta'(w), w^2\delta''(w); w)\!: w \in U\} \subset \Omega \Longrightarrow \delta(\mathrm{U}) \subset \Delta. \quad (4)$$

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then there is a conformal mapping σ of U onto Δ such that $\sigma(0)$=a. In this case , (4) can be written as:

$$\{\psi(\delta(w), z\delta'(w), w^2\delta''(w); w)\!: w \in U\} \subset \Omega \Longrightarrow \delta(\mathrm{U}) \subset \sigma(\mathrm{U}).$$

If Ω is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0)=\psi(a,0,0;0)$. Also , if the function $\psi(\delta(w), w\delta'(w), w^2\delta''(w); w)$ is analytic in U, then it;s can be written as:

$$\psi(\delta(w), w\delta'(w), w^2\delta''(w); w) \prec \mathrm{h(z)} \Longrightarrow \delta(\mathrm{z}) \prec \sigma(\mathrm{z}). \qquad (5)$$

Definition(2) [4]: Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If δ is analytic in U and satisfies the second –order differential subordination

$$\psi(\delta(w), w\delta'(w), w^2\delta''(w); w) \prec h(\mathrm{w}), \quad (6)$$

then $\delta$is called a solution of the differential subordination . The univalent function σ is called a dominant of the solutions of the differential subordination, relation (6) simply dominant if $\delta \prec \sigma$ for all δ satisfying (6).

A dominant $\breve{\sigma}$ is said to be best dominant if it satisfies $\breve{\sigma} \prec \sigma$ for all dominants σ of (6) .

Definition(3) [4]: Denote by Q the set of all functions σ that are injectiveand analytic on $\overline{U} \backslash E(\sigma)$, where

$$E(\sigma) = \{\xi \in \partial U: \lim_{z\to\xi} \sigma(z) = \infty\}, \qquad (7)$$

and are such that $\sigma'(\xi) \neq 0$ for $\xi \in \partial U \backslash E(\sigma)$.Further, let the subclass of Q for which $\sigma(0) = a$ be denoted by Q(a), Q(0)$\equiv Q_0$ and Q(1) $\equiv Q_1$, where Q_1= { q $\in$Q:q(0)= }.

Definition(4) [4]: Let Ω be a set in $\mathbb{C}$, $\sigma \in Q$ and n be positive integer. The class of admissible function $\Psi_n[\Omega, \sigma]$ consist of those functions

$\psi: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ which is satisfy the admissibility condition

$$\psi(r, s, t; w, \xi) \notin \Omega.$$

Whenever $r= \sigma(\xi), s = k\xi\sigma'(\xi)$ and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq k\ Re\left\{\frac{\xi\sigma''(\xi)}{\sigma'(\xi)} + 1\right\}, (8)$$

$w \in U, \xi \in \partial U \backslash E(\sigma), \xi \in \overline{U}$, and $k \geq n$. In particular case $\Psi_1[\Omega,, \sigma] = \Psi[\Omega, \sigma]$.

Definition(5) [4]: Let Ω be a set in $\mathbb{C}$, $\sigma \in \mu[a, n]$ and n be positive integer. The class of admissible function$\Psi'_n[\Omega, \sigma]$ consist of those functions

$\psi: \mathbb{C}^3 \times \overline{U} \times \overline{U} \to \mathbb{C}$ that satisfies the admissibility condition

$$\psi(r, s, t; \xi, \zeta) \notin \Omega.$$

Whenever $r= \sigma(w), s = \left(\frac{1}{m}\right) w\xi\sigma'(w)$ and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq \left(\frac{1}{m}\right) Re\left\{\frac{w\sigma''(w)}{\sigma'(w)} + 1\right\}, \quad (9)$$

$w \in U, \xi \in \partial U, \zeta \in \overline{U}$, and $m \geq n \geq 1$. In particular case for n=1

$$\Psi'_1[\Omega, \sigma] = \Psi'[\Omega, \sigma] \qquad (10).$$

Similar study is carried out by several authors , like Billing [1] Dihnggong and Liu[2], Oros [5,6], and Lupas [7,8]. Many authors studied different classes with essential problems such as Sarah A. AL-Ameedee ,Waggas Galib Atshan &Faez Ali AL-Maamori[9], Serkan Çakmak, Sibel Yalcçın, Şahsene Altınkaya[10], Waggas Galib Atshan and, Haneen Zaghir [11], Odeh Z. and Kassim A. Jassim[12], Odeh Z. and Kasim A. Jassim[13] and Odeh Z.and kassim a. Jassim[14]

Definition(6): Let Ω be a set in $\mathbb{C}$ and $\sigma \in Q_0 \cap \mu[0, \delta]$. The class of admissible functions $\Phi_k[\Omega, \sigma]$ consists of those functions $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\emptyset(u, v, w; w) \notin \Omega, \qquad (11)$$

whenever

$$u = \sigma(\zeta) \quad , v = kC_j^m(T)\zeta\sigma'(\zeta) + \left(1 - C_j^m(T)\right)\sigma(\zeta), (\lambda > -1),$$

and

$$\mathcal{R}e\left\{\frac{w-\left(C_j^m(b)^2-2C_j^m(b)+1\right)u-\left(2-C_j^m(b)\right)C_j^m(b)v}{C_j^m(b)^2\left(\frac{v-\left(1-C_j^m(b)\right)u}{C_j^m(b)}\right)}+1\right\} \geq kRe\left\{\frac{\zeta\sigma''(\zeta)}{\sigma'(\zeta)}+1\right\} \quad (12)$$

$w \in U, \zeta \in \partial U \setminus E(\sigma)$, and $k \geq 1$.

Theorem(1): Let $\emptyset \in \Phi_k[\Omega,\sigma]$. If $f \in \mathcal{H}$ satisfies

$$\left\{\emptyset\left(D_{m,b}^{\lambda}f(w), D_{m,b}^{\lambda+1}f(w), D_{m,b}^{\lambda+2}f(w); w\right)\right\} \subset \Omega, \quad (13)$$

then $D_{m,b}^{\lambda}f(w) \prec \sigma(w)$.

Proof: By using (2) and (3), we get the equivalent relation $D_{m,b}^{\lambda+1}f(w) = C_j^m(b)w\left(D_{m,b}^{\lambda}f(w)\right)' + (1-C_j^m(b))D_{m,b}^{\lambda}f(w)$. (14)

Assume that

$$\mathcal{F}(w) = D_{m,b}^{\lambda}f(w). \quad (15)$$

Then

$$D_{m,b}^{\lambda+1}f(w) = C_j^m(b)wF'(w) + (1-C_j^m(b))F(w) \quad (16)$$

Therefore,

$$D_{m,b}^{\lambda+2}f(w) = C_j^m(b)w\left(D_{m,b}^{\lambda+1}f(w)\right)' + (1-C_j^m(b))D_{m,b}^{\lambda+1}f(w) \quad (17)$$

then we have by (13),

$$\left(D_{m,b}^{\lambda+1}f(w)\right)' = C_j^m(b)wF''(w) + C_j^m(b)F'(w) + (1-C_j^m(b))F'(w). \quad (18)$$

So,

$$D_{m,b}^{\lambda+2}f(w) = C_j^m(b)^2w^2F''(w) + \left(2-C_j^m(b)\right)wC_j^m(b)F'(w) + \left(C_j^m(b)^2-2C_j^m(b)+1\right)F(w) \quad (19)$$

Let $u = r$, $v = C_j^m(b)s + \left(1-C_j^m(b)\right)r$

$$w = C_j^m(b)^2t + \left(2-C_j^m(b)\right)C_j^m(b)s + \left(C_j^m(b)^2-2C_j^m(b)+1\right)r$$

Assume that

$$\begin{aligned}\psi(r,s,t;w) &= \emptyset(u,v,w;\omega,\xi)\\ &= \emptyset\Big(r, C_j^m(b)s + \left(1-C_j^m(b)\right)r, C_j^m(b)^2t + \left(2-C_j^m(b)\right)C_j^m(b)s + \left(C_j^m(b)^2-2C_j^m(b)+1\right)r; w\Big).\end{aligned}$$

By using (15) and (19), we obtain

$$\psi(\mathcal{F}(w), w\mathcal{F}'(w), w^2\mathcal{F}''(w); w) = \emptyset\left(D_{m,T}^{\lambda}f(w), D_{m,T}^{\lambda+1}f(w), D_{m,T}^{\lambda+2}f(w); w\right). \quad (20)$$

Therefore, by making use (14), we get

$$\psi(\mathcal{F}(w), w\mathcal{F}'(w), z^2\mathcal{F}''(w); w) \in \Omega. \quad (21)$$

Also, by using

$$w = C_j^m(b)^2t + \left(2-C_j^m(b)\right)C_j^m(b)s + \left(C_j^m(b)^2-2C_j^m(b)+1\right)r$$

and by simple calculations, we get

$$\frac{w-\left(C_j^m(b)^2-2C_j^m(b)+1\right)u-\left(2-C_j^m(b)\right)C_j^m(b)v}{C_j^m(b)^2\left(\frac{v-\left(1-C_j^m(b)\right)u}{C_j^m(b)}\right)}+1 = \frac{t}{s}+1 \quad (22)$$

and the admissibility condition for $\emptyset \in \Phi_k[\Omega,\sigma]$ is equivalent to the admissibility condition for ψ then $\psi \in \Psi_n[\Omega,\sigma]$ *and* therefore $\mathcal{F}(w) \prec \sigma(w)$. Hence, we get $D_{m,T}^{\lambda}f(w) \prec \sigma(w)$.

If we assume that $\Omega \neq \mathbb{C}$ is a simply connected domain. So, $\Omega = h(U)$, for some conformal mapping h of U onto Ω. Assume the class is written as $\Phi_k[h,\sigma]$.Therefore, we conclude immediately the following theorem.

Theorem(2): Let $\emptyset \in \Phi_k[h,\sigma]$. If $f \in \mathcal{H}$ satisfies

$$\emptyset\left(D_{m,T}^{\lambda}f(w), D_{m,T}^{\lambda+1}f(w), D_{m,T}^{\lambda+2}f(w); w\right) \prec h(w), \quad (23)$$

then $D_{m,T}^{\lambda}f(w) \prec \sigma(w)$.

The next result is an extension of Theorem (1) to the case where the behavior of σ on ∂U is unknown.

Corollary(1): Let $\Omega \subset \mathbb{C}$, σ be univalent in U and $\sigma(0)=0$. Let $\emptyset \in \Phi_k[\Omega,\sigma_\rho]$ for some $\rho \in (0,1)$, where $\sigma_\rho(w) = \sigma(\rho w)$. If $f \in \mathcal{H}$ satisfies

$\emptyset\left(D_{m,b}^{\lambda} f(w), D_{m,b}^{\lambda+1} f(w), D_{m,b}^{\lambda+2} f(w); w\right) \in \Omega,$
(24)
then $D_{m,T}^{\lambda} f(w) \prec \sigma(w)$.

Theorem (3): Let h and σ be univalent in U, with $\sigma(0)=0$, $\sigma_{\rho}(w) = \sigma(\rho w)$ and $h_{\rho}(\omega) = h(\rho\omega)$. Let $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:
(1) $\emptyset \in \Phi_k[\Omega, \sigma_\rho]$ for some $\rho \in (0,1)$ or
(2) there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \sigma_\rho]$ for all $\rho \in (\rho_0, 1)$.
If $f \in \mathcal{H}$ satisfies (23) , then

$$D_{m,b}^{\lambda} f(w) \prec \sigma(w).$$

Proof: Case (1): By using Theorem (1),we get $D_{m,b}^{\lambda} f(w) \prec \sigma_\rho$.Since $\sigma_\rho(w) \prec \sigma(w)$ then we get the result.

Case (2): Assume that $\mathcal{F}(w) = D_{m,b}^{\lambda} f(w)$ and $F_\rho(w) = F(\rho w)$. So,

$$\emptyset\left(\mathcal{F}_\rho(w), z\mathcal{F}_\rho'(w), w^2\mathcal{F}_\rho''(w); \rho w\right) = \emptyset(\mathcal{F}(\rho w), \rho w\mathcal{F}'(\rho w), \rho^2 w^2 \mathcal{F}''(\rho w); \rho w) \in h_\rho(U).$$

By using Theorem (1) and associated with $\emptyset(\mathcal{F}(w), w\mathcal{F}'(w), w^2\mathcal{F}''(w); w(w)) \in \Omega$, where w is any function mapping from U onto U, with $w(w) = \rho w$, we obtain $\mathcal{F}_\rho(w) \prec \sigma_\rho(w)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $D_{m,b}^{\lambda} f(w) \prec \sigma(w)$.

The next theorem gives the best dominant of the differential subordination (23).

Theorem (4): Let h be univalent in U and let $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(\sigma(w), \sigma'(w), w^2\sigma''(w); w\right) = h(w)$$
(25)

has a solution σ with $\sigma(0)=0$ and satisfy one of the following conditions:
(1) $\sigma \in Q_0$ and $\emptyset \in \Phi_k[h, \sigma]$.
(2) σ is univalent in U and $\emptyset \in \Phi_k[h, \sigma_\rho]$ for some $\rho \in (0,1)$.
(3) σ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \sigma_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{H}$ satisfies (23),then $D_{m,b}^{\lambda} f(w) \prec q(w)$ and σ is the best dominant.

Proof: By using Theorem (2) and Theorem (3) , we get that σ is a dominant of (23). Since σ satisfies (25), it is also a solution of (23) and therefore σ will be dominant by all dominants of (23). Hence , σ is the best dominant of (23).

Definition(7): Let Ω be a set in $\mathbb{C}$ and $\sigma \in Q_0 \cap \mu_0$. The class of admissible functions $\Phi_{k,1}[\Omega, \sigma]$ consists of those functions $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\emptyset(u, v, w; w) \notin \Omega,$$

whenever

$$u = \sigma(\zeta) \quad , \quad v = kC_j^m(b)\zeta\sigma'(\zeta) + \sigma(z) \, , (\lambda > -1),$$

and

$$\mathcal{R}e\left\{\frac{(w-u)}{vC_j^m(b)^2} - \frac{(C_j^m(b)+2)}{C_j^m(b)} + 1\right\} \geq kRe\left\{\frac{\zeta\sigma''(\zeta)}{\sigma'(\zeta)} + 1\right\} \quad (26)$$

Theorem(5): Let $\emptyset \in \Phi_{k,1}[\Omega, \sigma]$. If $f \in \mathcal{H}$ satisfies

$$\left\{\emptyset\left(\frac{D_{m,b}^{\lambda} f(w)}{w}, \frac{D_{m,b}^{\lambda+1} f(w)}{w}, \frac{D_{m,b}^{\lambda+2} f(w)}{w}; w\right)\right\} \subset \Omega,$$
(27)

then $\frac{D_{m,b}^{\lambda} f(w)}{w} \prec \sigma(w)$.

Proof: Let the analytic function $\mathcal{F}$ in U be defined by

$$\mathcal{F}(w) = \frac{D_{m,T}^{\lambda} f(w)}{w} \quad (28).$$

Then by (13) and (28), we get

$$D_{m,b}^{\lambda+1} f(w) = C_j^m(b) w\left(D_{m,b}^{\lambda} f(w)\right)' + (1 - C_j^m(b)) D_{m,b}^{\lambda} f(w) = C_j^m(b) w(wF(w))' + (1 - C_j^m(b)) wF(w) = C_j^m(b) w\left(wF'(w) + F(w)\right) + (1 - C_j^m(b)) wF(w)$$

$$= C_j^m(b)w^2F'(w) + C_j^m(b)wF(w) + wF(w) - C_j^m(b)wF(w)$$

This implies that

$$\frac{D_{m,T}^{\lambda+1}f(w)}{w} = C_j^m(b)\big(wF'(w) + F(w)\big) + \Big(1 - C_j^m(b)\Big)F(w)$$

$$= C_j^m(b)wF'(w) + C_j^m(b)F(w) + F(w) - C_j^m(b)F(w)$$

$$= C_j^m(b)wF'(w) + F(w) \qquad (29)$$

Also, we have by (17)

$$D_{m,b}^{\lambda+2}f(w) = C_j^m(b)w\left(D_{m,b}^{\lambda+1}f(w)\right)' + (1 - C_j^m(b))D_{m,b}^{\lambda+1}f(w)$$

$$= C_j^m(b)w\left(D_{m,b}^{\lambda+1}f(w)\right)' + (1 - C_j^m(b))D_{m,b}^{\lambda+1}f(w)$$

$$= C_j^m(b)w\left(C_j^m(b)w^2F'(w) + C_j^m(b)wF(w) + wF(w) - C_j^m(b)wF(w)\right)' + (1 - C_j^m(b))(C_j^m(b)w^2F'(w) + C_j^m(b)wF(w) + wF(w) - C_j^m(b)wF(w))$$

$$= C_j^m(b)w\left(C_j^m(b)w^2F'(w) + wF(w)\right)' + (1 - C_j^m(b))(C_j^m(b)w^2F'(w) + wF(w))$$

$$= C_j^m(b)w\big(C_j^m(b)w^2F''(w) + F'(w)2wC_j^m(b) + wF'(w) + F(w)\big) + ((C_j^m(b)w^2F'(w) + wF(w) - C_j^m(T)^2(b)w^2F'(w) - wF(w)C_j^m(b))$$

$$= w\left(C_j^m(b)^2w^2F''(w) + F'(w)2wC_j^m(b)^2 + wF'(w)C_j^m(b) + F(w)C_j^m(b)\right) + ((C_j^m(b)w^2F'(w) + wF(w) - C_j^m(b)^2(b)w^2F'(w) - wF(w)C_j^m(b))$$

Then

$$\frac{D_{m,b}^{\lambda+2}f(w)}{w} = C_j^m(b)^2w^2F''(w) + F'(w)2wC_j^m(b)^2 + wF'(w)C_j^m(b) + F(w)C_j^m(b) + (C_j^m(b)wF'(w) + F(w) - C_j^m(b)^2(b)wF'(w) - F(w)C_j^m(b) \quad (30)$$

So, let define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by U=r , v=$C_j^m(b)s + r$

w=$C_j^m(b)^2t + 2sC_j^m(b)^2 + sC_j^m(b) + rC_j^m(b) + C_j^m(b)s + r - C_j^m(b)^2(b)s - rC_j^m(b)$

$$= C_j^m(b)^2t + sC_j^m(b)^2 + 2sC_j^m(b) + r$$

$$= C_j^m(b)^2t + sC_j^m(b)(C_j^m(b) + 2) + r$$

and by simple calculations ,we get

$$\frac{(w-u)}{vC_j^m(b)^2} - \frac{\left(C_j^m(b) + 2\right)}{C_j^m(b)} + 1 = \frac{t}{s} + 1$$

Theorem(6): Let $\emptyset \in \Phi_k[h,q]$. If $f \in \mathcal{H}$ satisfies

$$\emptyset\left(\frac{D_{m,b}^{\lambda}f(w)}{w}, \frac{D_{m,b}^{\lambda+1}f(w)}{w}, \frac{D_{m,b}^{\lambda+2}f(w)}{w}; w\right) \prec h(w), \qquad (31)$$

then $\frac{D_{m,b}^{\lambda}f(w)}{w} \prec \sigma(w)$.

The next result is an extension of Theorem (1) to the case where the behavior of σ on ∂U is unknown.

Corollary(2): Let $\Omega \subset \mathbb{C}$, σ be univalent in U and $\sigma(0)=0$. Let $\emptyset \in \Phi_k[\Omega, \sigma_\rho]$ for some $\rho \in (0,1)$, where $\sigma_\rho(w) = \sigma(\rho w)$. If $f \in \mathcal{H}$ satisfies

$$\emptyset\left(\frac{D_{m,b}^{\lambda}f(w)}{w}, \frac{D_{m,b}^{\lambda+1}f(w)}{w}, \frac{D_{m,b}^{\lambda+2}f(w)}{w}; w\right) \in \Omega, \quad (32)$$

then $\frac{D_{m,b}^{\lambda}f(w)}{w} \prec \sigma(w)$.

Proof:The proof is completed by using Theorem(1) .

Theorem (7): Let h and σ be univalent in U , with $\sigma(0)=0$, $\sigma_\rho(w) = \sigma(\rho w)$ and $h_\rho(\omega) =$

$h(\rho\omega)$. Let $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:

(1) $\emptyset \in \Phi_k[\Omega, \sigma_\rho]$ for some $\rho \in (0,1)$ or

(2) there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \sigma_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{H}$ satisfies (23) , then

$$\frac{D^{\lambda}_{m,b} f(w)}{w} \prec \sigma(w).$$

<u>Proof:</u> Case (1): By using Theorem (1),we get $D^{\lambda}_{m,T} f(w) \prec \sigma_\rho$.Since $\sigma_\rho(w) \prec q(w)$ then we get the result.

Case (2): Assume that $\mathcal{F}(w) = D^{\lambda}_{m,T} f(w)$ and $\mathcal{F}_\rho(w) = \mathcal{F}(\rho w)$. So,

$$\emptyset(\mathcal{F}_\rho(w), w\mathcal{F}'_\rho(w), w^2\mathcal{F}''_\rho(w); \rho w)$$
$$= \emptyset(\mathcal{F}(\rho w), \rho w\mathcal{F}'(\rho w), \rho^2 w^2 \mathcal{F}''(\rho w); \rho w)$$
$$\in h_\rho(U).$$

By using Theorem (1) and associated with $\emptyset(\mathcal{F}(w), w\mathcal{F}'(w), w^2\mathcal{F}''(w); w(w))$ $\in \Omega$, where w is any function mapping from U onto U, with $w(w) = \rho w$, we obtain $\mathcal{F}_\rho(w) \prec \sigma_\rho(w)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $\frac{D^{\lambda}_{m,b} f(w)}{w} \prec \sigma(w)$.

The next theorem gives the best dominant of the differential subordination (23).

<u>Theorem (8):</u> Let h be univalent in U and let $\emptyset: \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset(\sigma(w), \sigma'(w), w^2\sigma''(w); w) = h(w) \quad (33)$$

has a solution σ with $\sigma(0)=0$ and satisfy one of the following conditions:

(1) $\sigma \in Q_0$ and $\emptyset \in \Phi_k[h, \sigma]$.

(2) σ is univalent in U and $\emptyset \in \Phi_k[h, \sigma_\rho]$ for some $\rho \in (0,1)$.

(3) σ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \sigma_\rho]$,

for all $\rho \in (\rho_0, 1)$.If $f \in \mathcal{H}$ satisfies (23),then $\frac{D^{\lambda}_{m,b} f(w)}{w} \prec \sigma(w)$ and σ is the best dominant.

REFERENCE

1. Billing S.S. , An application of differential subordination for starlikness of analytic functions , Int. J. Open problems complex analysis , 2(3)(2010) , 221-229.
2. Dinggong Y. and J. L. Liu, On a class of analytic functions involving Rusceweyh derivative , Bull , Korean Math. Soc. , 39(1)(2002), 123-131.
3. Frasin B., A new differential operator of analytic functions involving binomial series, Bol.Soc. Paran. Mat. 38(5) (2020), 205-213.
4. Miller S.S. and P.T.Mocanu , Differential Subordinations :Theory and Applications, Series on Monographs and Text books in pure and Applied Mathematics Vol. 225 , Marcel Dekker , New York and Basel, (2000).
5. Oros G.I., Strong differential superordination. Acta Univ. Apulensis, Mat.-Inform. 19(2009), 101-106.
6. Oros G.I., Strong differential subordination. Turk. J. Math. 33, (2009) 249-257.
7. Lapus A.A., Certain differential subordinations using a generalized salagean operator and Ruscheweyh operator, Internationl Journal of Fractional Calculus and applied Analysis, 13(4)(2010), 355-360.
8. Lapus A.A., A note on differential subordinations using Salagean and Ruscheweyhoperators , Romi J. , 6(1)(2010), 4 pages.

- 9. Sarah A. AL-Ameedee ,Waggas Galib Atshan &Faez Ali AL-Maamori, On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics 23(4),(2020):803-809.
- Sibel Yalcçın, Şahsene Altınkaya , On a subclass of harmonic univalent functions based on subordination, Journal Theory and Applications of Mathematics & Computer Science, 7(2), (2017), 51–62.

11. Waggas Galib Atshan and, Haneen Zaghir Hassan,Differential Sandwich Results For Univalent Functions, AL-Qadisiyah Journal Of Pure Science,.25, (1), (2020) 55–59.

12. Odeh Z. and Kassim A. Jassim,On the class of multivalent analytic functions defined by differential operator for derivative of first

order,Journal of Al-Qadisiyah for Computer Science and Mathematics, , 11(1) (2019), 80-86.

13. Odeh Z. and Kasim A. Jassim,Subordination for certain analytic multivalent functions defined by using differential operator, IOP Conf. Series: Journal of Physics: Conf. Series1234 (2019) 012110.

14. Odeh Z.and kassim a. Jassim, Multivalent harmonic starlike functions of complex order defined by a linear, operator, Sci.int.(lahore), 31(1), (2019) , 11-18 .