On classes of meromorphic functions with fixed point and fixed second and finitely many coeeficients defined by q -Derivative

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Abstract: In this paper we consider the class $M_q S^*(\zeta,\alpha,c)$ which consisting of meromorphic univalent functions with a fixed point ζ in $U^* := \{z \: : \: z \in C, 0 < |z| < 1\}$ and with fixed second positive coefficient. The aim of the present paper is to drive several interesting properties as coefficient estimates, distortion theorems, radii of starlikeness and convexity and closure theorems of $f(z)$ in the class $M_q S^*(\zeta,\alpha,c)$. The results are generalized to families with finitely many fixed coefficients.

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1. Introduction

The applications of q -calculus are new area in the last 25 years. This great interest is due to its application in various branches of mathematics and physics. The first applications of q -calculus was made by Jackson [9, 10]. He developed q -derivative and q -integral in a systematic. Aral and Gupta [5, 1, 7] defined the q -analogue of Baskakov Durrmeyer operator which based on q -analogue of beta function. The authors studied approximation and geometric properties of these q -operators in some subclasses of analytic functions in compact disk. Studies on quantum groups have played an important role in defining geometrical interpretation of q -analysis. It also suggests a relation between integrable systems and q -analysis.

Another important q -generalization of complex operators is q -Picard and q -Gauss-Weierstrass singular integral operators discussed in [1, 2, 3]. Other q -analogues of differential operators have been introduced in [11].

Let ζ be a fixed point in the punctured unit disk $U^* = \{z \in C, 0 < |z| < 1\} = U /\{0\}$.

Denote by $M(\zeta)$ be the class of meromorphic functions of the form

$$f(z) = \frac{1}{z-\zeta} + \sum_{k=1}^{\infty} a_k (z-\zeta)^k, \quad z \neq \zeta.$$
$$(1.1)$$

A function $f(z)$ of the form (1.1) is in the class of meromorphic starlike of order α $(0 \leq \alpha < 1)$ denoted by $S(\zeta,\alpha)$, if

$$R\left\{-\frac{(z-\zeta)f^{'}(z)}{f(z)}\right\} > \alpha, \ (z-\zeta) \in U$$

and is in the class of meromorphic convex of order α $(0 \le \alpha < 1)$ denoted by $K(\zeta, \alpha)$, if

$$R\left\{ -\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)} \right) \right\} > \alpha, \quad (z-\zeta) \in U.$$

We recall some concepts of q-calculus, all details about q-calculus used in this paper can be found in [4, 8], for $n \in N$, the q-number is defined as follows:

$$[k]_q = \frac{1-q^k}{1-q}, \quad 0 < q < 1. \tag{1.2}$$

Hence, $[k]q$ can be expressed as a geometric series $\sum\limits_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. As $q \oplus 1$, $[k] \to k$. The q-derivative of a function f(z) is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (q \neq 1, z \neq 0)$$

and $D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. For a function $g(z) = z^k$ we found that

$$D_q(g(z)) = D_q z^k = \frac{(1-q^k)}{(1-q)} z^{k-1} = [k]_q z^{k-1}, \quad k \in N$$

and

$$\lim_{q \to 1^-} D_q(g(z)) = \lim_{q \to 1^-} [k]_q z^{k-1} = kz^{k-1} = g'(z),$$

where g' is the ordinary derivative. The q-Jackson definite integral of the function $f(z)$ is defined by

$$\int_0^z f(t)d_q t = (1-q)z \sum_{n=0}^{\infty} f(zq^n)q^n, \quad z \in C.$$

Definition 1: *A function $f(z)$ given by (1.1) is in the class $M_q S(\zeta, \alpha)$ of meromorphically starlike functions of order ζ in U^* if it satisfies*

$$Re\left\{ -\frac{q(z-\zeta)D_q f(z)}{f(z)} \right\} > \alpha \quad ((z-\zeta) \in U, 0 < q < 1, 0 \le \alpha < \tag{1.3}$$

Let $M_p(\zeta)$ denote the subclass of $M(\zeta)$ consisting of functions of the form

$$f(z) = \frac{1}{z-\zeta} + \sum_{k=1}^{\infty} a_k(z-\zeta)^k, \quad (a_k \ge 0). \tag{1.4}$$

Also, we define the class $M_q S^*(\zeta, \alpha)$ by

$$M_q S^*(\zeta, \alpha) = M_q S(\zeta, \alpha) \cap M_p(\zeta).$$

2. Coefficient Estimates

Theorem 1: *Let the function $f \in M_p(\zeta)$ be given by (1.4). Then $f \in M_q S^*(\zeta, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \left(q[k]_q + \alpha \right) a_k \le 1 - \alpha \quad (a_k \ge 0), \tag{2.1}$$

where $0 < q < 1$ and $0 \le \alpha < 1$. The result is sharp.

Proof: Let $f(z) \in M_q S^*(\zeta, \alpha)$. Then in view of (1.3), we get

$$Re\left\{ -\frac{q(z-\zeta)D_q f(z) - \alpha f(z)}{f(z)} \right\} \ge 0.$$

A simple computation yields

$$Re\left\{ \frac{\frac{1-\alpha}{z-\zeta} - \sum\limits_{k=1}^{\infty} (q[k]_q + \alpha)a_k(z-\zeta)^k}{\frac{1}{z-\zeta} + \sum\limits_{k=1}^{\infty} a_k(z-\zeta)^k} \right\} \ge 0.$$

The above condition must hold for all values of $(z-\zeta)$ in U if we choose $(z-\zeta)$ to be the real and $(z-\zeta) \to 1^-$ we get

$$\frac{(1-\alpha) - \sum\limits_{k=1}^{\infty} (q[k]_q + \alpha)a_k}{1 + \sum\limits_{k=1}^{\infty} a_k} \ge 0$$

$$(1-\alpha) - \sum_{k=1}^{\infty} (q[k]_q + \alpha)a_k \ge 0$$

$$\sum_{k=1}^{\infty}(q[k]_q + \alpha)a_k \le (1-\alpha).$$

Conversely, by applying the hypothesis (2.1) and choosing $|z - \zeta| = r < 1$ we find that

$$\text{Re}\left\{-\frac{q(z-\zeta)D_q f(z)}{f(z)}\right\} \ge \frac{\alpha(1+\sum_{k=1}^{\infty} a_k)}{1+\sum_{k=1}^{\infty} a_k} = \alpha.$$

Then $f \in M_q S^*(\zeta, \alpha)$.

Lemma 1: *Let function $f(z)$ given by (1.4) be in the class $M_q S^*(\zeta, \alpha)$, which satisfies*

$$a_k \le \frac{1-\alpha}{q[k]_q + \alpha}.$$

Hence we can take

$$a_1 \le \frac{1-\alpha}{q+\alpha}.$$

Let $M_q S^*(\zeta, \alpha, c)$ denote the class of functions $f(z)$ in $M_q S^*(\zeta, \alpha)$ of the form

$$f(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty} a_k (z-\zeta)^k \quad (a_k \ge 0)$$

(2.2)

where $0 < q < 1$, $0 \le \alpha < 1$ and $0 < c < 1$.

Theorem 2: *Let the function $f(z)$ be defined by (2.2). Then $f(z) \in M_q S^*(\zeta, \alpha, c)$ if and only if*

$$\sum_{k=2}^{\infty}\left(q[k]_q + \alpha\right)a_k \le (1-\alpha)(1-c).$$

(2.3)

The result is sharp.

Proof: Putting

$$a_1 \le \frac{c(1-\alpha)}{q+\alpha}, \qquad 0 \le c \le 1,$$

(2.4)

in (2.1), and simplifying we get the result.

The result is sharp for the function

$$f(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \frac{(1-c)(1-\alpha)}{(q[k]_q + \alpha)}(z-\zeta)^k, \quad (k \ge$$

(2.5)

Corollary: *Let the function $f(z)$ defined by (2.2) be in the class $M_q S^*(\zeta, \alpha, c)$. Then*

$$a_k \le \frac{(1-c)(1-\alpha)}{(q[k]_q + \alpha)} \quad (k \ge 2).$$

(2.6)

The result is sharp for the function $f(z)$ given by (2.5).

3. Closure Theorems

Theorem 3: *The class $M_q S^*(\zeta, \alpha, c)$ is closed under convex linear combination.*

Proof: Let the function $f(z)$ be defined by (2.2). Define the function $h(z)$ by

$$h(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty} b_k (z-\zeta)^k \quad (b_k \ge 0).$$

(3.1)

Suppose that $f(z)$ and $h(z)$ are in the class $M_q S^*(\zeta, \alpha, c)$, it is enough to prove that the function

$$H(z) = \lambda f(z) + (1-\lambda)h(z) \quad (0 \le \lambda \le 1)$$

(3.2)

also be in the class $M_q S^*(\zeta, \alpha, c)$.

Since

$$H(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty}\{\lambda a_k + (1-\lambda)b_k\}(z-\zeta)^k$$

(3.3)

we obtain that

$$\sum_{k=2}^{\infty}\left(q[k]_q + \alpha\right)\{\lambda a_k + (1-\lambda)b_k\} \le (1-\alpha)(1-c)$$

(3.4)

by returning to the Theorem 2. Hence $H(z) \in M_q S^*(\zeta, \alpha, c)$. This clearly

completes the proof of the Theorem.

Theorem 4: *Let the functions*

$$f_j(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty} a_{k,j}(z-\zeta)^k, \quad (a_{k,j} \geq 0),$$

(3.5)

be in the class $M_q S^*(\zeta, \alpha, c)$ for every

$j = 1, 2, ..., m$. *Then the function* $F(z)$

defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0)$$

(3.6)

is also in the class $M_q S^*(\zeta, \alpha, c)$, *where*

$$\sum_{j=1}^{m} d_j = 1.$$

(3.7)

Proof: From (3.5), (3.6) and (3.7), we have

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} d_j a_{k,j} \right)(z-\zeta)^k.$$

(3.8)

Since $f_j(z) \in M_q S^*(\zeta, \alpha, c)$ for every j=1,2,...,m, in view of Theorem 2, we get

$$\sum_{k=2}^{\infty} \left(q[k]_q + \alpha \right) a_{k,j} \leq (1-\alpha)(1-c), \quad j = 1, 2, ..., m.$$

(3.9)

Thus we obtain

$$\sum_{k=2}^{\infty} \left(q[k]_q + \alpha \right) \left(\sum_{j=1}^{m} d_j a_{k,j} \right) = \sum_{j=1}^{m} d_j \left(\sum_{k=2}^{\infty} \left(q[k]_q + \alpha \right) a_{k,j} \right) \leq (1-\alpha)(1-c),$$

which implies that $F(z) \in M_q S^*(\zeta, \alpha, c)$.

Theorem 5: *Let*

$$f_1(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta)$$

(3.10)

and

$$f_k(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \frac{(1-c)(1-\alpha)}{(q[k]_q + \alpha)}(z-\zeta)^k$$

(3.11)

(for $k \geq 2$. Then the function $f(z)$ is in the class $M_q S^*(\zeta, \alpha, c)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

(3.12)

where $\lambda_k \geq 0$ $(k \geq 1)$ and

$$\sum_{k=1}^{\infty} \lambda_k = 1.$$

(3.13)

Proof : *We suppose that the function $f(z)$ can be expressed in the form (3.12). Then from (3.10), (3.11) and (3.13), we have*

$$f(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty} \frac{\lambda_k(1-c)(1-\alpha)}{(q[k]_q + \alpha)}(z-\zeta)^k,$$

since

$$\sum_{k=2}^{\infty} \frac{\lambda_k(1-c)(1-\alpha)}{(q[k]_q + \alpha)} \cdot (q[k]_q + \alpha)$$

$$= (1-c)(1-\alpha) \sum_{k=2}^{\infty} \lambda_k$$

$$= (1-c)(1-\alpha)(1-\lambda)$$

$$\leq (1-c)(1-\alpha).$$

Then $f(z) \in M_q S^*(\zeta, \alpha, c)$. Conversely, assuming that $f(z)$ defined by (2.2) be in the class $M_q S^*(\zeta, \alpha, c)$ which satisfies (2.6), we obtain

$$a_k \leq \frac{(1-c)(1-\alpha)}{(q[k]_q + \alpha)} \quad (k \geq 2).$$

Setting

$$\lambda_k = \frac{(q[k]_q + \alpha)}{(1-c)(1-\alpha)} a_k \quad (k \geq 2),$$

and

$$\lambda = 1 - \sum_{k=2}^{\infty} \lambda_k,$$

this gives (3.12). This compelets the proof of the Theorem.

Corollary 2: *The extreme points of the class* $M_q S^* (\zeta, \alpha, c)$ *are the functions* $f_k(z)$ $(k \geq 1)$ *given by (3.10) and (3.11) in Theorem 5.*

4. Growth and Distortion Theorems

Theorem 6: *If the function* $f(z)$ *defined by (2.2) is in the class* $M_q S^*(\zeta, \alpha, c)$ *for* $0 \boxminus |z \approx| \; \blacksquare r \boxminus 1,$ *then we have*

$$\frac{1}{r} - \frac{(1-\alpha)c}{(q+\alpha)} r - \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r^2 \leq | f(z) |$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)c}{(q+\alpha)} r + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r^2$$

with equality for

$$f_2(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)}(z-\zeta)^2.$$

Proof : Suppose that $f(z)$ is in the class $M_q S^*(\zeta, \alpha, c)$. By Theorem 2, we have

$$a_k \leq \frac{(1-\alpha)(1-c)}{(q[k]_q + \alpha)}, \qquad k \geq 2.$$

Thus, for $0 < |z-\zeta| = r < 1$

$$|f(z)| \leq \frac{1}{|z-\zeta|} + \frac{(1-\alpha)c}{(q+\alpha)}|z-\zeta| + \sum_{k=2}^{\infty} a_k |z-\zeta|^k, \quad |z-\zeta| = r$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)c}{(q+\alpha)} r + r^2 \sum_{k=2}^{\infty} a_k$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)c}{(q+\alpha)} r + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r^2$$

and

$$|f(z)| \geq \frac{1}{|z-\zeta|} - \frac{(1-\alpha)c}{(q+\alpha)}|z-\zeta| - \sum_{k=2}^{\infty} a_k |z-\zeta|^k, \quad (|z-\zeta| = r)$$

$$\geq \frac{1}{r} - \frac{(1-\alpha)c}{(q+\alpha)} r - r^2 \sum_{k=2}^{\infty} a_k$$

$$\geq \frac{1}{r} - \frac{(1-\alpha)c}{(q+\alpha)} r - \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r^2.$$

The proof is complete.

Theorem 7: *If the function* $f(z)$ *defined by (2.2) is in the class* $M_q S^*(\zeta, \alpha, c)$ *for* $0 < |z-\zeta| = r < 1$ *, then we have*

$$\frac{1}{r^2} - \frac{(1-\alpha)c}{(q+\alpha)} - \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r \leq |f'(z)|$$

$$\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{(q+\alpha)} + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r$$

with equality for

$$f_2(z) = \frac{1}{z-\zeta} + \frac{(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)}(z-\zeta)^2.$$

Proof : From Theorem 2, it follows that

$$ka_k \leq \frac{k(1-\alpha)(1-c)}{(q[k]_q + \alpha)}, \qquad k \geq 2.$$

Thus, for $0 < |z-\zeta| = r < 1$, we obtain

$$|f'(z)| \leq \left| \frac{-1}{(z-\zeta)^2} \right| + \frac{(1-\alpha)c}{(q+\alpha)} + \sum_{k=2}^{\infty} ka_k |z-\zeta|^{k-1}, \qquad (|z-\zeta|=r)$$

$$\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{(q+\alpha)} + r \sum_{k=2}^{\infty} ka_k$$

$$|z-\zeta| = r \leq \frac{1}{r^2} + \frac{(1-\alpha)c}{(q+\alpha)} + \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)} r.$$

and

$$|f'(z)| \geq \left| \frac{-1}{(z-\zeta)^2} \right| - \frac{(1-\alpha)c}{(q+\alpha)} - \sum_{k=2}^{\infty} ka_k |z-\zeta|^{k-1}, \qquad (|z-\zeta|=r)$$

$$\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{(q+\alpha)} - r\sum_{k=2}^{\infty} ka_k$$

$$\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{(q+\alpha)} - \frac{(1-\alpha)(1-c)}{(q^2+q+\alpha)}r.$$

The proof of the Theorem is complete.

5. Radii of Starlikeness and Convexity

Theorem 8: *Let the function $f(z)$ defined by (2.2) is in the class $M_q S^*(\zeta,\alpha,c)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho \leq 1)$ in the disk $|z-\zeta| < r_1(\zeta,\alpha,c,\rho)$, where $r_1(\zeta,\alpha,c,\rho)$ is the largest value for which*

$$\frac{(1+\rho)(1-\alpha)c}{(q+\alpha)}r^2 + \frac{(1-\alpha)(1-c)(k+\rho)}{(q[k]_q+\alpha)}r^{k+1} \leq (1-\rho)$$

(5.1)

for $k \geq 2$. The result is sharp for function $f(z)$ given by (2.5).

Proof :   It suffices to show that

$$\left| \frac{(z-\zeta)f^{'}(z)}{f(z)} + 1 \right| \leq (1-\rho)$$

for $|z-\zeta| < r_1(\zeta,\alpha,c,\rho)$. We have

$$\left| \frac{(z-\zeta)f^{'}(z)}{f(z)} + 1 \right| = \left| \frac{\frac{2(1-\alpha)c}{(q+\alpha)}(z-\zeta) + \sum_{k=2}^{\infty}(k+1)a_k(z-\zeta)^k}{\frac{1}{z-\zeta} + \frac{(1-\alpha)c}{q+\alpha}(z-\zeta) + \sum_{k=2}^{\infty}a_k(z-\zeta)^k} \right|$$

(5.2)

Hence (5.2) holds true if

$$\frac{2(1-\alpha)c}{(q+\alpha)}r^2 + \sum_{k=2}^{\infty}(k+1)a_k r^{k+1}$$

$$\leq (1-\rho)\left(1 + \frac{(1-\alpha)c}{(q+\alpha)}r^2 + \sum_{k=2}^{\infty}a_k r^{k+1} \right)$$

(5.3)

or

$$\frac{(1+\rho)(1-\alpha)c}{(q+\alpha)}r^2 + \sum_{k=2}^{\infty}(k+\rho)a_k r^{k+1} \leq (1-\rho)$$

(5.4)

and it follows that from (2.3), we may take

$$a_k \leq \frac{(1-c)(1-\alpha)}{(q[k]_q+\alpha)}\lambda_k, \quad k \geq 2$$

where $\lambda_k \geq 0$ and $\sum_{k=2}^{\infty}\lambda_k \leq 1$.

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which $\frac{(k+\rho)}{(q[k]_q+\alpha)}r^{k+1}$ is maximal. Then it follows that

$$\sum_{k=2}^{\infty}(k+\rho)a_k r^{k+1} \leq \frac{(1-c)(1-\alpha)(k_0+\rho)}{(q[k_0]_q+\alpha)}r^{k_0+1}.$$

(5.5)

Then $f(z)$ is starlike of order ρ in $0 < |z-\zeta| < r_1(\zeta,\alpha,c,\rho)$ provided that

$$\frac{(1+\rho)(1-\alpha)c}{(q+\alpha)}r^2 + \frac{(1-\alpha)(1-c)(k_0+\rho)}{(q[k_0]_q+\alpha)}r^{k_0+1} \leq (1-\rho)$$

(5.6)

we find the value $r_0 = r_0(\zeta,\alpha,c,\rho)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{(1+\rho)(1-\alpha)c}{(q+\alpha)}r_0^2 + \frac{(1-\alpha)(1-c)(k_0+\rho)}{(q[k_0]_q+\alpha)}r^{k_0+1} = (1-\rho).$$

(5.7)

Then this value is the radius of starlikeness of order ζ for function f belongs to class $M_q S^*(\zeta,\alpha,c)$.

Theorem 9:   *Let the function $f(z)$ defined by (2.2) is in the class $M_q S^*(\zeta,\alpha,c)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho \leq 1)$ in the disk $|z-\zeta| < r_2(\zeta,\alpha,c,\rho)$, where $r_2(\zeta,\alpha,c,\rho)$ is the largest value for which*

$$\frac{(1+\rho)(1-\alpha)c}{(q+\alpha)}r^2 + \frac{k(k+\rho)(1-\alpha)(1-c)}{(q[k]_q+\alpha)}r^{k+1} \leq (1-\rho)$$

(5.8)

for $k \geq 2$. The result is sharp for function $f(z)$ given by (2.5).

Proof :   It suffices to show that

$$\left|\frac{(z-\zeta)f''(z)}{f'(z)}+2\right|\le(1-\rho)$$

for $|z-\zeta|<r_2$ with the aid of Theorem 2. Then by using the same technique in the proof of Theorem 8 we can complete the proof.

6. The Class $M_qS^*(\zeta,\alpha,c_n)$

In addition to fixing the second coefficient, we can fix finitely many coefficients. Let $M_qS^*(\zeta,\alpha,c_n)$ denote the class of functions in $M_qS^*(\zeta,\alpha)$ of the form

$$f(z)=\frac{1}{z-\zeta}+\sum_{i=1}^{n}\frac{c_i(1-\alpha)}{(q[i]_q+\alpha)}(z-\zeta)^i+\sum_{k=n+1}^{\infty}a_k(z-\zeta)^k$$

(6.1)

where $0\le\sum_{i=1}^{n}c_i=c\le1$. Note that

$$M_qS^*(\zeta,\alpha,c_1)=M_qS^*(\zeta,\alpha,c).$$

7. properties of $M_qS^*(\zeta,\alpha,c_n)$

Theorem 10: *(coefficient estimates) A function f⬦ defined by (6.1) is in $M_qS^*(\zeta,\alpha,c_n)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(q[k]_q+\alpha\right)a_k\le(1-\alpha)(1-\sum_{i=1}^{n}c_i),$$

where $0\le\sum_{i=1}^{n}c_i=c\le1$ *and* $0\diamond c_i\diamond1$.

The result is sharp.

Proof : Putting

$$a_i\le\frac{c_i(1-\alpha)}{(q[i]_q+\alpha)},\qquad i=1,2,3,...,n,$$

in (2.1), and simplifying we get the result. The result is sharp for the function

$$f(z)=\frac{1}{z-\zeta}+\sum_{i=1}^{n}\frac{(1-\alpha)c_i}{(q[i]_q+\alpha)}(z-\zeta)^i+\frac{(1-\sum_{i=1}^{n}c_i)(1-\alpha)}{(q[k]_q+\alpha)}(z-\zeta)^k$$

(7.1)

Corollary 3: *Let the function* $f(z)$ *defined by (6.1) be in the class* $M_qS^*(\zeta,\alpha,c_n)$. *Then*

$$a_k\le\frac{(1-\sum_{i=1}^{n}c_i)(1-\alpha)}{(q[k]_q+\alpha)},\qquad(k\ge n+1).$$

(7.2)

The result is sharp for the function $f(z)$ *given by (7.1).*

Theorem 11: *The class* $M_qS^*(\zeta,\alpha,c_n)$ *is closed under convex linear combination.*

Proof : Let the function $f(z)$ be defined by (6.1). Define the function $h(z)$ by

$$h(z)=\frac{1}{z-\zeta}+\sum_{i=1}^{n}\frac{c_i(1-\alpha)}{(q[i]_q+\alpha)}(z-\zeta)^i+\sum_{k=n+1}^{\infty}b_k(z-\zeta)^k.$$

Suppose that $f(z)$ and $h(z)$ are in the class $M_qS^*(\zeta,\alpha,c_n)$, it is enough to prove that the function

$$H(z)=\lambda f(z)+(1-\lambda)h(z)\qquad(0\le\lambda\le1)$$

also be in the class $M_qS^*(\zeta,\alpha,c_n)$. Since

$$H(z)=\frac{1}{z-\zeta}+\sum_{i=1}^{n}\frac{c_i(1-\alpha)}{(q[i]_q+\alpha)}(z-\zeta)^i+\sum_{k=n+1}^{\infty}\{\lambda a_k+(1-\lambda)b_k\}$$

we obtain that

$$\sum_{k=n+1}^{\infty}\left(q[k]_q+\alpha\right)\{\lambda a_k+(1-\lambda)b_k\}\le(1-\alpha)(1-\sum_{i=1}^{n}c_i)$$

by returning to the Theorem 10. Hence $H(z)\in M_qS^*(\zeta,\alpha,c_n)$. This clearly completes the proof of the Theorem.

Theorem 12: *Let the functions*

$$f_j(z)=\frac{1}{z-\zeta}+\sum_{i=1}^{n}\frac{c_i(1-\alpha)}{(q[i]_q+\alpha)}(z-\zeta)^i+\sum_{k=n+1}^{\infty}a_{k,j}(z-\zeta)^k,$$

(7.3)

be in the class $M_q S^*(\zeta, \alpha, c_n)$ *for every*

$j = 1, 2, ..., m$. *Then the function* $F(z)$

defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0)$$

(7.4)

is also in the class $M_q S^*(\zeta, \alpha, c_n)$, *where*

$$\sum_{j=1}^{m} d_j = 1.$$

(7.5)

Proof :   From (7.3), (7.4) and (7.5) we have

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1 - \alpha)}{(q[i]_q + \alpha)}(z - \zeta)^i + \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{m} d_j a_{k,j} \right)(z - \zeta)^k.$$

Since $f_j(z) \in M_q S^*(\zeta, \alpha, c_n)$ for every

j=1,2,...,m, in view of Theorem 10, we get

$$\sum_{k=n+1}^{\infty} \left(q[k]_q + \alpha \right) a_{k,j} \leq (1 - \alpha)(1 - \sum_{i=1}^{n} c_i), \quad j = 1, 2, ..., m.$$

Thus we obtain

$$\sum_{k=n+1}^{\infty} \left(q[k]_q + \alpha \right) \left(\sum_{j=1}^{m} d_j a_{k,j} \right) = \sum_{j=1}^{m} d_j \left(\sum_{k=n+1}^{\infty} \left(q[k]_q + \alpha \right) a_{k,j} \right) \lesseqgtr (1 - \sum_{i=1}^{n} c_i)(1 - \alpha).$$

which implies that $F(z) \in M_q S^*(\zeta, \alpha, c_n)$.

Theorem 13: *Let*

$$f_n(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1 - \alpha)}{(q[i]_q + \alpha)}(z - \zeta)^i$$

(7.6)

and

$$f_k(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1 - \alpha)}{(q[i]_q + \alpha)}(z - \zeta)^i + \frac{(1 - \sum_{i=1}^{n} c_i)(1 - \alpha)}{(q[k]_q + \alpha)}(z - \zeta)^k$$

(7.7)

for $k \geq n+1$. *Then the function* $f(z)$ *is in the class* $M_q S^*(\zeta, \alpha, c_n)$ *if and only if it*

can be expressed in the form

$$f(z) = \sum_{k=n}^{\infty} \lambda_k f_k(z),$$

(7.8)

where $\lambda_k \geq 0 \quad (k \geq n)$ *and*

$$\sum_{k=n}^{\infty} \lambda_k = 1.$$

(7.9)

Proof : *We suppose that the function* $f(z)$ *can be expressed in the form (7.8). Then from (7.6) ,(7.7) and (7.9) we have*

$$f(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1 - \alpha)}{(q[i]_q + \alpha)}(z - \zeta)^i + \sum_{k=n+1}^{\infty} \frac{\lambda_k(1 - \sum_{i=1}^{n} c_i)(1 - \alpha)}{(q[k]_q + \alpha)}$$

$$\sum_{k=n+1}^{\infty} \frac{\lambda_k(1 - \sum_{i=1}^{n} c_i)(1 - \alpha)}{(q[k]_q + \alpha)} . (q[k]_q + \alpha)$$

$$= (1 - \sum_{i=1}^{n} c_i)(1 - \alpha) \sum_{k=n+1}^{\infty} \lambda_k$$

$$= (1 - \sum_{i=1}^{n} c_i)(1 - \alpha)(1 - \lambda)$$

$$\lesseqgtr (1 - \sum_{i=1}^{n} c_i)(1 - \alpha).$$

Then $f(z) \in M_q S^*(\zeta, \alpha, c_n)$.

Conversely, assuming that $f(z)$ *defined by (6.1) be in the class* $M_q S^*(\zeta, \alpha, c_n)$ *which satisfies (7.2)，we obtain*

$$a_k \leq \frac{(1 - \sum_{i=1}^{n} c_i)(1 - \alpha)}{(q[k]_q + \alpha)} \quad (k \geq n+1).$$

Setting

$$\lambda_k = \frac{(q[k]_q + \alpha)}{(z - \zeta)^k(1 - \sum_{i=1}^{n} c_i)(1 - \alpha)} a_k \quad (k \geq n+1),$$

and

$$\lambda_n = 1 - \sum_{k=n+1}^{\infty} \lambda_k.$$

Corollary 4: *The extreme points of the class* $M_q S^*(\zeta, \alpha, c_n)$ *are*

$$f_n(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1-\alpha)}{(q[k]_q + \alpha)}(z-\zeta)^i$$

and

$$f_k(z) = \frac{1}{z - \zeta} + \sum_{i=1}^{n} \frac{c_i(1-\alpha)}{(q[i]_q + \alpha)}(z-\zeta)^i + \frac{(1-c)(1-\alpha)}{(q[k]_q + \alpha)}(z-\zeta)^k, \quad k \geq n+1.$$

Theorem 14: *Let the function* $f(z)$ *defined by (6.1) is in the class* $M_q S^*(\zeta, \alpha, c_k)$. *Then* $f(z)$ *is convex of order* ρ $(0 \leq \rho \leq 1)$ *in the disk* $|z - \zeta| < r_3(\zeta, \alpha, c, \rho)$, *where* $r_3(\zeta, \alpha, c, \rho)$ *is the largest value for which*

$$\sum_{i=1}^{n} \frac{i(i+\rho)(1-\alpha)c_i}{(q[i]_q + \alpha)} r^{i+1} + \frac{k(k+\rho)(1-\alpha)(1-\sum_{i=1}^{n}c_i)}{((q[k]_q + \alpha)} r^{k+1} \leq (1-\rho)$$

(7.10)

for $k \geq n+1$. *The result is sharp for function* $f(z)$ *given by (7.1).*

Proof : It suffices to show that

$$\left| \frac{(z-\zeta)f^{''}(z)}{f^{'}(z)} + 2 \right| \leq (1-\rho)$$

for $|z - \zeta| < r_2$ with the aid of Theorem 2. Then by using the same technique in the proof of Theorem 8 we can complete the proof.

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