Some Chaos on Graph Maps

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Abstract: Let G be a graph and $f: G \longrightarrow G$ be continuous function, we study some types of chaotic functions on a graph and find the relation between them. We also introduce a new type of chaos defined on a graph called strongly chaotic and characterization generically chaotic and densely chaotic on graph maps.

Keywords: Chaos, Graph map, dynamical system.

1. Introduction

The purpose of this paper is to study some notions of chaos for graph map and find the relation between them. Grzegorz .H et al. [5], studied two definitions of chaos on graph map,the first Auslander and Yorke: a map f is chaotic if it is transitive and sensitive to initial conditions. The second is Devaney chaos: a map f is chaotic if it is transitive and sensitive to initial conditions, and the set of periodic points of f is dense, prove that Auslander and Yorke chaos implies Devaney chaos, and weak mixing implies mixing for graph map. Miyazawa [10], showed that Devaney chaos and ω-chaos are equivalent for graph maps. Roman and Michal [11] proved that the distributional chaos and positive topological entropy are equivalent for continuous graph maps. Ruette and Snoha[4] studied the Li-Yorke chaos for graph maps and proved the existence of a scrambled pair implies Li-Yorke chaos.

A pair (X, f) is a discrete dynamical system if X is a topological space and $f: X \to X$ is continuous. The orbit of a point $x \in X$, denoted by $O_f(x)$, is the set $O_f(x) = \{f^n(x) \setminus n = 0,1,2....\}$ $f^n(x)$ can be considered as the new position of x after n units of time, where $f^n = f \circ f \circ \cdots \circ f$ is the composition of f taken n-times. The point x is a periodic point if there exists an integer $n \geq 1$ such that $f^n(x) = x$, and the set of all periodic points denoted by $P(f)$. A map f is transitive if for every pair of non-empty open sets U & V in X there is a positive integer n, such that $f^n(U) \cap V \neq \phi$, and f is said to be totally transitive if f^n is transitive for all integers $n > 1$, note that if $n \in \mathrm{N}$ and f^n is transitive then f is transitive. The converse is not true in general. f is said to be mixing if for all non-empty open sets U, V there exists N, such that for all $n \geq N, f^n(U) \cap V \neq \emptyset$, and f is weakly topologically mixing, or just weakly mixing if the Cartesian product $f \times f$ is transitive. This is equivalent to saying that, if U_i and V_i are non-empty open sets for each $i \leq m$, then there is a $n > 1$ such that $f^n(U_i) \cap V_i \neq \emptyset$ for each $i \leq m$. For $\delta > 0$, f is said to be sensitive dependence on initial conditions (SDIC) if, for every point $x \in X$ and $\varepsilon > 0$, there exists a point $y \in X$ with $d(x, y) < \varepsilon$ and $n \in \mathrm{N}$ such that $d(f^n(x), f^n(y)) > \delta$.

A topological graph is non-degenerate compact connected metric space G containing a finite subset V, whose points are called vertices, such that each connected component of G / V is the disjoint union of finite number of open subsets of G, called edge, with property that each edges is homeomorphic to an open interval of real line and it's boundary consists of at most two points which are called the vertices. Note that each graph is locally connected. A circuit is a subset of G homeomorphic to a circle and any graph without circuits is called tree. A graph map is a dynamical system on a graph, that is a continuous map $f: G \longrightarrow G$, where G is a graph.

2. Transitive Graph Maps

In this paper, we shall consider transitive maps on graphs which are not trees. Such graph are not contractible to one point, therefore, it may be exist transitive maps on these graph having no periodic points. In fact according to [1], transitive graph map without periodic points may be defined on connected graphs only if these graph are homeomorphic to the circle.

Theorem (2-1)[1]: Let G be a graph and let $f \in C(G,G)$ be transitive map without periodic points. Then f is totally transitive, G is the circle and f is conjugate to an irrational rotation (in particular f has zero topological entropy).

Theorem (2-2)[1]: Let G be a graph and let $f \in C(G,G)$ be a transitive map. Then either the set of periodic points of f is dense in G and f has positive topological entropy, or G is the circle and f is conjugate to an irrational rotation of the circle. Now, we can classify the transitive map on a graph in two classes: map with periodic point (dense periodic points, positive topological entropy), and map without periodic points (f is totally transitive, zero topological entropy), theorem (2-3), completes the characterization of totally transitive graph map.

Theorem (2-3)[1]: Let G be graph and let $f \in C(G,G)$ be transitive. Then the following statements are equivalent:

(i) f is totally transitive.

(ii) f is topologically mixing.

(iii) The set of periodic points is cofinite in$\mathbb{N}$

It is well know that mixing implies weak mixing, the following results discuss the relationship between mixing and weak mixing and totally transitive on graph map.

Theorem (2-4)[5]: If f is a weakly mixing graph map, then f is mixing.

Theorem (2-5)[6]: A totally transitive map with a dense set of periodic points is weakly mixing.

Corollary (2-6)[6]: If $f$is totally transitive and has a dense periodic points ,f^n is weakly mixing for each $n \geq 1$.

Now, we can prove the following corollary:

Corollary (2-7): If $f \in C(G,G)$ is totally transitive graph map with periodic points then f is mixing.

Proof: since the f is totally transitive map, so f is transitive and has periodic points, according to theorem (2-2), f has dense periodic points. Hence by theorem (2-5) f is weakly mixing. By theorem (2-4) f is mixing.

Total transitive is reassuring property to discretize a continuous dynamical system. As we shall see the total transitive has a strong relation with the notion below.

Definition(2-8)[12]: Let (X,d) be a compact metric space, a map$f: X \longrightarrow X$ is said to verify the specification property if for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$, such that for any collection of points $x_1, x_2, x_3, \dots, x_k$ with $k \geq 2$, for any collection of integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$,such that $a_i - b_{i-1} \geq M(\epsilon)$ and for any $p \in \mathbb{N}$ such that $p \geq M(\epsilon) + b_k - a_1$,there exists a point$y \in X$ such that $f^p(y) = y$ and $d(f^n(y), f^n(x_i)) \leq \epsilon$ for each $n, a_i \leq n \leq b_i, 1 \leq i \leq k$. when $k = 2$ we say f verifies the weak specification property.

The relation between the specification property and the totally transitive map can be found in the following theorem.

Theorem (2-9)[12]: Let G be graph and let $f \in C(G,G)$ be transitive. Then the following statements are equivalent:

(a) f is totally transitive

(b) f verifies the specification property.

(c)f verifies the weak specification property.

(d) f is topologically mixing.

We recall the definition of strong transitive [8], a map$f: G \longrightarrow G$ is called strongly transitive if for every non-empty open set $J \subset G$,there exists an n such that $\bigcup_{k=0}^{n} f^k(J) = G$. From the above definition every strong transitive is transitive, but the converse is not true in generally, we can see an examples in [8].

Theorem (2-10)[8]: let $f: G \longrightarrow G$ be a graph map with $\#Fix(f^k) < \infty$ for each $k \geq 1$. If f is transitive, then it is strongly transitive.

From the above theorem it is easy to prove the following corollary:

Corollary (2-11): Every totally transitive graph map is strongly transitive.

3. Chaotic Map:

One of the most popular chaotic mappings is Devaney chaos introduced by Devaney in [4]. Another widely recognized indicator of chaotic behavior of the system is positively of topological entropy (see[10]).

According to the Devaney's definition of chaos [4], a map $f: X \longrightarrow X$ is chaotic if it transitive, periodic points are dense, and it's sensitive dependent on initial conditions. However, in graph G if $f: G \longrightarrow G$ is transitive with periodic point then the second and third of condition is redundant (see[9]).

One can consider stronger notions of chaos by replacing transitivity by strongly transitive. We will go to the extreme and consider the strongest of those notions. we will say that is strong Devaney chaos if it strongly transitive, dense periodic point, sensitive dependent on initial conditions. Note that every strongly transitive map is transitive, so we can prove the following theorem.

Theorem (3-1): Let $f: G \longrightarrow G$ be a graph map, if f is strongly Devaney chaos, then f is Devaney chaos.

The converse of above the theorem is not true generally, for example see (example 4 in [8]). On the other hand, for continuous map f of the graph in to itself Miyazawa [10] introduced the following notion ω-chaos, and showed that f is ω-chaos if and only if has positive entropy. A subset S of X is an ω-scrambled set for f if, for any $x, y \in S$ with $x \neq y$, the following condition hold[10]:

$(\omega 1)$ $\omega(x,f) \backslash \omega(y,f)$ is uncountable.

$(\omega 2)$ $\omega(x,f) \cap \omega(y,f) \neq \emptyset$.

$(\omega 3)$ $\omega(x,f) \not\subset per(f)$.

Where the set $\omega(x,f)$ is an ω-limit set of a point $x \in X$. We say that f is ω-chaos if there exists an uncountable ω-scrambled set of f. The following theorem have been proved in [10].

Theorem (3-2): Let f be continuous map of a graph into itself. The following conditions are equivalent:

(i) f has positive topological entropy.

(ii) f is ω-chaos

(iii) f is chaotic in the sense of Devaney.

We can state and prove the following corollary:

Corollary (3-3): let $f \in C(G,G)$ be transitive map with periodic points then f is ω-chaos

Proof: since f transitive map with periodic points then by theorem (2-2), f has positive topological entropy, so f is ω-chaos by theorem(3-2).

The notion of distributional chaos was introduced in [12]. In same paper the authors show that the presence of distributional chaos, and positive topological entropy are equivalent for continuous interval maps. Roman Hric and Michal [11] showed the distributional chaos and positive topological entropy are equivalent for continuous graph maps in general.

Theorem (3-4): Let G be a graph and $f \in C(G,G)$ then the following conditions are equivalent:

(i) $h(f) > 0$,

(ii) f^n has a horseshoe ,for some $n \in \mathbb{N}$.

(iii) f has a basic set.

(iv) f is distributional chaotic.

Chaotic in the sense of Li-Yorke means that the system has an uncountable scrambled set S in which arbitrary $(x,y) \in S \times S: x \neq y$ is a Li-Yorke pair. A pair of points $\{x,y\} \subset X$ is said to be Li-Yorke pair if

$\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$ $\lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) > 0$ and

Denotes the set of Li-Yorke pair of f by $LY(f) = \{(x,y) \in X^2 : \lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) > 0, \lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0\}$. In [13] the following theorem had proved.

Theorem (3-5): Let G be a graph and $f \in C(G,G)$, if f has a scrambled pair, then it has a cantor δ- scrambled set for some $\delta > 0$.

For graph maps, positive topological entropy equivalent with the existence of infinite ω-limit set containing a periodic points[13]. The system (G,f) is Li-Yorke chaotic if it has an uncountable scrambled set, in fact the existence of a cantor scrambled set. Now we can prove the following theorem:

Theorem (3-6): let $f \in C(G,G)$ be transitive map with periodic points then f is Li-Yorke chaotic.

Proof: suppose f is transitive map with periodic points then, by theorem(2-2), $h(f) > 0$ and by[7,TheoremB], there are closed intervals $J, K \subseteq G$ with disjoint interiors, and $n \in \mathbb{N}$ such that $J \cup K \subseteq f^n(J) \cap f^n(K)$, then[3, pp35-37], f has an infinite ω-limit set containing periodic point, then there exist scrambled pairs in the system by theorem(3-5), f has a cantor δ- scrambled set for some $\delta > 0$ this implies f is Li-Yorke chaotic.

From the above results, we can see that when the graph map is transitive and has periodic point then f has positive topological entropy and all type of chaos are equivalent.

Theorem (3-7):): Let $f \in C(G,G)$ be transitive map with periodic points then the following conditions are equivalent:

(i) f has positive topological entropy.
(ii) f is chaotic in the sense of Devaney.
(iii) f is strongly chaotic.
(iv) f is ω-chaos.
(v) f is distributional chaotic.
(vi) f is Li-Yorke chaotic.

4. Characterization of chaos:

A function$f$ from a real compact interval I into itself is called generically chaotic[14] ,if the set of all points (x, y) for which:
$\lim_{n\to\infty} \sup|f^n(x), f^n(y)| > 0$ and $\lim_{n\to\infty} \inf|f^n(x), f^n(y)| = 0$, are residual in $I \times I$, and we say f is densely chaotic if this set is dense in $I \times I$.

Now, we may generalize the generically chaotic and densely chaotic on graph maps:

Definition (4-1): A function $f \in C(G,G)$ is called generically chaotic if the set of All points (x, y) for which:
$\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$ and $\lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) > 0$, is residual in $G \times G$, and we say f is densely chaotic if this set is dense in $G \times G$. If the set of all $x, y \in G$ such that:
$\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$ and $\lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) \geq \delta$, is residual in $G \times G$, then f is called generically δ-chaotic and we say f is densely δ-chaotic if this set is dense in$G \times G$, where $\delta > 0$.

It is clear the generic chaos implies dense chaos but not conversely (see Example 3.1 in [14]); we can prove the following theorem:

Theorem (4-2): Let $f \in C(G,G)$ be transitive map with periodic points then f is generically chaotic, moreover f is densely chaotic.

Proof: Since f is transitive map with periodic point then by theorem (2-1), f is totally transitive and by theorem (2-9), f is wealky mixing.

So ,$(G \times G, f \times f)$ is transitive $\Longrightarrow f \times f$ has dense orbit in $G \times G$, then set of dense orbit is dense G_δ-set. By compactness, there exist $x_1, x_2 \in G$ such that $d(x_1, x_2) = \delta$. If (x, y) belong to the set of dense orbit, then the exist two subsequences (n_i) and (m_i) such that:
$\big(f^{n_i}(x), f^{n_i}(y)\big) \to (x_1, x_2)$ and $\big(f^{m_i}(x), f^{m_i}(y)\big) \to (x_1, x_1)$ when $i \longrightarrow \infty$.

Consequently $\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$ and $\lim_{n\to\infty} \sup d\big(f^n(x), f^n(y)\big) > 0$, that is all points in the set of dense orbit is residual and dense in $G \times G$. Thus f is generically chaotic and densely chaotic.

Definition (4-3)[15]: A dynamical system is positive expansive if there is a constant $\epsilon > 0$ with the property that, for any $x \neq y$, there is $n \in \mathbb{N}$ for which $\big(f^n(x), f^n(y)\big) \geq \epsilon$.

Theorem (4-4): Let $f: G \longrightarrow G$ be continuous graph map. If f is densely δ-chaos then f is positive expansive.

Proof : Since f is densely δ-chaos, let $x \in G$ and $U \subseteq G$ be open set, such that $x \in U$ then there exist (x_1, x_2) in $U \times U$,and the exist sequence (n_k), such that $\lim_{k\to\infty} d\big(f^{n_k}(x_1), f^{n_k}(x_2)\big) \geq \delta$, for any $\epsilon < \frac{\delta}{2}$, we have
$\lim_{k\to\infty} d\big(f^{n_k}(x), f^{n_k}(x_1)\big) + \lim_{k\to\infty} d\big(f^{n_k}(x), f^{n_k}(x_2)\big) \geq \lim_{k\to\infty} d\big(f^{n_k}(x_1), f^{n_k}(x_2)\big)$,

then $\lim_{k\to\infty} d\big(f^{n_k}(x), f^{n_k}(x_i)\big) \geq \epsilon$, where $i = 1,2$.

That mean f is positive expansive.

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