

On Lyapunov- Schmidt Reduction to Reduce Differential Algebraic Equation

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Abstract: In this paper , we transformation singularity differential – Algebraic equation to an ordinary differential equations by use Lyapunov– Schmidt reduction and constructing Lyapunov functions depending on Reiss and Geiss method.

Keywords: Reiss and Geiss method, singularity, Lyapunov– Schmidt reduction.

1. Introduction

The word singularity is used pervasively in mathematics. Even within a given research field, this term often has different meanings. Differential –algebraic equation theory is not an exception; the expression singular system has been used to mean at least differential algebraic system, or higher index differential algebraic system. Roughly speaking singular differential– algebraic equations or more precisely, singular points of differential– algebraic equations will be locally defined as those for which the assumptions supporting an index notion conditions in [1, 2].

The importance of studying singularity comes since if the system of equations is regular at all its points then the system will be just an explicit ordinary differential equations system [3] (an index zero differential– algebraic equation), hence there should be some singular points, such problems arise naturally in a variety of applications, e.g., electrical networks, constraint mechanical systems of rigid bodies, chemical reaction kinetics.

2. Lyapunov– Schmidt Reduction [4]

We discuss the case when the semi– explicit differential algebraic equation formula is singular.

Let

$$X^\circ = f(x, y) \tag{1}$$

$$0 = g(x, y)$$

where $f, g: R^n \times R^m \rightarrow R^n \times R^m$ has a critical point at $(0, 0)$ and the condition.

$$\text{rank } D_y g(x, y) = m - 1 \tag{2}$$

holds. Let $D_y g(x, y) = B$ then $\text{rank } B = m - 1$

Choose complements vector spaces (H) and (N) to Kernel B and range B respectively. Then:

$$\text{Ker } B \oplus H = R^m \tag{3}$$

$$N \oplus \text{range } B = R^m \tag{4}$$

From (3) we conclude that $\dim H = m - 1$, and from (4) that $\dim N = 1$. Define a projection $E: R^m \rightarrow \text{range } B$ and the complementary projection

$$I - E : R^m \rightarrow N$$

Such that expanded to an equivalent pair of equation:

$$\begin{aligned} X^\circ &= f(x, y) \\ E g(x, y) &= 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} X^\circ &= f(x, y) \\ (I - E) g(x, y) &= 0 \end{aligned} \tag{6}$$

Because of this splitting any vector $y \in R^m$ can be decomposed in the form $y = v \oplus w$, where $v \in \text{Ker } B$ and $w \in H$. Then (5) can be written as:

$$\begin{aligned} X^\circ &= f(x, v \oplus w) \\ E g(x, v \oplus w) &= 0 \end{aligned} \tag{7}$$

The second part of the equation (7) can be considered as a map:

$$\emptyset : R^n \times Ker B \times H \times R^v \rightarrow range B$$

where

$$\emptyset(x, v, w) = E g(x, v \oplus w)$$

Since E acts as the identity map on range B then we have:

$$\left(\frac{\partial E g(x, v \oplus w)}{\partial w} \right)_{(0,0)} = EB = B$$

and since

$$B : H \rightarrow range B$$

Has a full rank at (0, 0), it follows from the implicit function theorem [5] that the second equation of (5) can be solved uniquely for w near (0, 0) i. e.,

w = W(x, y) where:

$$W : R^n \times Ker B \times R^v \text{ satisfies}$$

$$E g(x, v \oplus w(x, v, \lambda)) = 0, w(0, 0, 0) = 0 \tag{8}$$

From the second equation of (7) and from differential– algebraic equation (1) we get the reduce differential– algebraic equations.

$$X^\circ = F(x, y) \tag{9}$$

$$0 = G(x, v)$$

Where $F, G : R^n \times Ker B \rightarrow R^n \times N$ are defined by

$$G(x, v) = (I - E) g(x, v + W(x, v))$$

$$\tag{10}$$

$$F(x, v) = f(x, v + W(x, v))$$

3. Construction of Lyapunov Functions for Reduce Differential– Algebraic Equation

In this section, we will construction Lyapunov functions by use Reiss and Geiss method on reduce differential– algebraic equation.

3.1. Reiss and Geiss method [6]:

Reiss and Geiss in 1963 proposed a method which is applicable to the system

$$X^\circ = F(x)$$

This method is a simple application of integration by parts to the problem of constructing a V function.

Since V° need only to be semidefinite, the simplest form that one can choose is the square of a state variable $V^\circ = -x_n^2$. If this choice does

not prove to be suitable then one may choose $V_1^\circ = -X_{n-1}^2$ and V° as follows:

$$V^\circ = V^\circ + \alpha V_1^\circ = -(X_n^2 + \alpha X_{n-1}^2),$$

where (α) is arbitrary constant.

4. A New Method for Construction of Lyapunov Function

Now, we introduce the new method for construction of Lyapunov function for non-singularity differential algebraic equations:

Consider non singularity differential algebraic equation, non-singularity differential algebraic equation Now consider below equation:

$$X_1^\circ = -x_1 + x_2 - x_2^3$$

$$X_2^\circ = x_1 - 4x_2$$

Assuming that $V^\circ = -x_2^2$, which is equivalent to ($-V^\circ = x_2^2$) and by integration an using the first equation, we get:

$$-V^\circ = \int_0^t x_2^2 dt$$

$$= \int_0^t x_2 (x_1 + x_1 + x_2^3) dt$$

$$= \int_0^t x_2 x_1 dt + \int_0^t x_1 x_2 dt + \int_0^t x_2^4 dt$$

Hence, integrating the first integration by parts, getting.

$$-V^\circ = x_1 x_2 - \int_0^t x_1^2 dt + 4 \int_0^t x_1 x_2 dt$$

$$+ \int_0^t x_1 x_2 dt + \int_0^t x_2^4 dt$$

$$= x_1 x_2 - \int_0^t x_1^2 dt$$

$$+ 4 \int_0^t x_1 (x_1$$

$$+ x_1 + x_2^3) dt$$

$$+ \int_0^t (x_2^\circ + 4x_2) x_2 dt + \int_0^t x_2^4 dt$$

$$\begin{aligned}
 &= x_1 x_2 - \int_0^t x_1^2 dt \\
 &\quad + 4 \int_0^t x_1 x_1^\circ dt + 4 \int_0^t x_1^2 dt \\
 &\quad + 4 \int_0^t x_1 x_2^3 dt + \int_0^t (x_2 x_2^\circ dt) \\
 &\quad + 4 \int_0^t x_2^2 dt + \int_0^t x_2^4 dt \\
 &= x_1 x_2 - \int_0^t x_1^2 dt + 2 x_1^2 + 4 \int_0^t x_1^2 dt \\
 &\quad + 4 \int_0^t (x_2^\circ + 4 x_2) x_2^3 dt + \frac{x_2^2}{2} \\
 &\quad + 4 \int_0^t x_2^2 dt + \int_0^t x_2^4 dt \\
 &= x_1 x_2 - \int_0^t x_1^2 dt + 2 x_1^2 + 4 \int_0^t x_1^2 dt + x_2^4 \\
 &\quad + 16 \int_0^t x_2^4 dt + \frac{x_2^2}{2} + 4 \int_0^t x_2^2 dt \\
 &\quad + \int_0^t x_2^4 dt \\
 &= 2x_1^2 + x_1 x_2 + \frac{x_2^2}{2} + x_2^4 - \int_0^t x_1^2 dt + 4 \int_0^t x_1^2 dt \\
 &\quad + 16 \int_0^t x_2^4 dt + 4 \int_0^t x_2^2 dt \\
 &\quad + \int_0^t x_2^4 dt
 \end{aligned}$$

Hence:

$$\begin{aligned}
 -V_\circ &= 2x_1^2 + x_1 x_2 + \frac{x_2^2}{2} + x_2^4 + 3 \int_0^t x_1^2 dt \\
 &\quad + 17 \int_0^t x_2^4 dt + 4 \int_0^t x_2^2 dt
 \end{aligned}$$

Let:

$$\begin{aligned}
 V &= -V_\circ - \left\{ 3 \int_0^t x_1^2 dt + 17 \int_0^t x_2^4 dt \right. \\
 &\quad \left. + 4 \int_0^t x_2^2 dt \right\} \\
 &= 2x_1^2 + x_1 x_2 + \frac{x_2^2}{2} + x_2^4
 \end{aligned}$$

and:

$$\begin{aligned}
 \dot{V} &= -V_\circ - 3x_1^2 - 17x_2^4 - 4x_2^2 \\
 &= x_2^2 - 3x_1^2 - 17x_2^4 - 4x_2^2
 \end{aligned}$$

Then:

$$\dot{V} = -3x_1^2 - 3x_2^2 - 17x_2^4 < 0 \text{ for } (x_1, x_2) \neq 0$$

Therefore, from V and \dot{V} above it follows that the zero solution is asymptotically stable.

References:

- [1] Rabier, P. J.; and Werner, G.; 1990, "Rheinbddd Ageneral Existance and Uniqueness theorem for Implicite Differential Algebraic Equation", University of Pittsburgh.
- [2] Reich, S; 1990, "On a Geometrical Interperation of Differential Algebraic Equation", Potsdam university.
- [3] Michael, H.; 1995, "Regularizations of Differential Algebraic Equations", Revisited Humboldt– University, Berlin.
- [4] Kamal, H. Y.; 2007, "Lyapunov– Schmidt Reduction of Differential Algebraic Equation", J. Basrah Researches (sciences) Vol. 33.
- [5] Jaber, H. A.; 1994, "Stability of Ordinary and Delay Differential Equations", Saddam University, Iraq, Baghdad.
- [6] Ibrahim, H.; 2013, "Bifurcation in Differential Algebraic Equations without Reduction with Application on Circuit Simulation", MSc thesis, Thi-Qar university.