

On Harmonic Univalent Functions Involving q -Poisson Distribution Series

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Abstract: Lately, the q - derivative operator has been used to investigate several subclasses of harmonic functions in different ways with different perspectives by many researchers and many interesting results were obtained. The q - derivative operator are also used to construct some subclasses of harmonic functions. In this paper, we define involving of q -Poisson distribution three harmonic functions and we aim to find the conditions for these functions to belong to the subclasses of q -starlike and q -convex harmonic univalent functions.

Keywords: Complex harmonic functions, univalent functions, q -calculus, q -starlike functions, q -convex functions.

1. Introduction

Let H denote the class of continuous complex-valued harmonic functions which are harmonic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U . A function harmonic in U may be written as $f = h + \bar{g}$, where h and g are analytic in U . We call h the analytic part and g co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see [4]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k. \quad (1)$$

Let SH denote the class of functions $f = h + \bar{g}$ which are harmonic, univalent and sense-preserving in U for which $h(0) = h'(0) =$

$1 = 0 = g(0)$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. The subclass SH^0 of SH consist of all functions in SH which have the additional property $b_1 = 0$. Clunie and Sheil-Small [4] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses.

The theory of quantum calculus known as q -calculus is equivalent to traditional infinitesimal calculus without the notion of limits. Throughout this article, we will use basic notations and definitions of the q -calculus as follows: Let $0 < q < 1$. For any non-negative integer k , the q -integer number k , denoted by $[k]_q$ is

$$[k]_q = \frac{1 - q^k}{1 - q}. \quad (k = 1, 2, 3, \dots), [0]_q = 0.$$

Similarly, the q -differential operator of a function f analytic in U is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \in U.$$

One can easily show that $D_q f(z) \rightarrow f(z)$ as $q \rightarrow 1^-$. Also, clearly $\lim_{q \rightarrow 1^-} [k]_q = k$. For details on q -calculus, one can refer to [7].

In 1990, Ismail et. al. [6] used q -calculus, in the theory of analytic univalent functions by defining a class of complex valued functions that are analytic on the open unit disk U with the normalizations $f(0) = 0, f'(0) = 1$, and $|f(qz)| \leq |f(z)|$ on U for every $q, q \in (0,1)$. Motivated by these authors, several researches used the theory of analytic and harmonic univalent functions and q -calculus; for example see [1] and [10]. The q -difference operator of analytic functions h and g given by (1) are by definition, given as follows [7]

$$\begin{aligned} D_q h(z) &= \begin{cases} \frac{h(z) - h(qz)}{(1-q)z} & ; z \neq 0 \\ h'(0) & ; z = 0 \end{cases} \quad \text{and} \\ D_q g(z) &= \begin{cases} \frac{g(z) - g(qz)}{(1-q)z} & ; z \neq 0 \\ g'(0) & ; z = 0 \end{cases}. \end{aligned} \quad (2)$$

Thus, for the function h and g of the form (1), we have

$$D_q h(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}$$

and

$$D_q g(z) = \sum_{k=1}^{\infty} [k]_q b_k z^{k-1}.$$

A harmonic function $f = h + \bar{g}$ defined by (1) is said to be q -harmonic, locally univalent and sense-preserving in U denoted by SH_q , if and only if the second dilatation w_q satisfies the condition

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1 \quad (3)$$

where $0 < q < 1$ and $z \in U$. Note that as $q \rightarrow 1^-$, SH_q reduces to the family SH (see [1]).

Denote by $SH_q^*(\alpha)$ the subclass of SH_q consisting of functions f of the form (1) that satisfy the condition

$$\operatorname{Re} \left[\frac{z D_q h(z) - \overline{z D_q g(z)}}{h(z) + \overline{g(z)}} \right] > \alpha \quad (4)$$

and denote by $KH_q(\alpha)$ the subclass of SH_q consisting of functions $f = h + \bar{g}$ of the form (1) that satisfy the condition

$$\operatorname{Re} \left[\frac{z D_q (z D_q h(z)) - \overline{z D_q (z D_q g(z))}}{z D_q h(z) - \overline{z D_q g(z)}} \right] > \alpha \quad (5)$$

where is $0 < q < 1, z \in U, 0 \leq \alpha < 1, D_q h(z)$ and $D_q g(z)$ are defined by (2) (see for details [10]). We will call respectively, q -starlike and q -convex harmonic functions of order α .

Define $TSH_q^*(\alpha) = SH_q^*(\alpha) \cap T^1$ and $TKH_q(\alpha) = KH_q(\alpha) \cap T^2$ where $T^m, (m = 1, 2)$ consisting of the functions $f = h + \bar{g}$ in H so that $h(z)$ and $g(z)$ are of the form

$$\begin{aligned} h(z) &= z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) \\ &= (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| z^k, \\ &(m = 1, 2). \end{aligned} \quad (6)$$

By suitably specializing the parameters, the classes $SH_q^*(\alpha)$ and $KH_q(\alpha)$ reduce to the various subclasses of harmonic univalent functions. Such as,

- (i) $SH_q^*(\alpha) = SH^*(\alpha)$ for $q \rightarrow 1^-$ ([8]),
- (ii) $KH_q(\alpha) = KH(\alpha)$ for $q \rightarrow 1^-$ ([9]),

- (iii) $SH_q^*(0) = SH^*$ for $q \rightarrow 1^-$ ([16]),
 (iv) $KH_q(0) = KH$ for $q \rightarrow 1^-$ ([17]).

The elementary distributions such as the Poisson, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view see ([2], [3], [5], [11], [12], [13], [14], [15]).

A variable x is said to q -Poisson Distribution if it takes the values $0, 1, 2, 3, \dots$ with probabilities e_q^{-r} , $\frac{r}{[1]_q!} e_q^{-r}$, $\frac{r^2}{[2]_q!} e_q^{-r}$, \dots , respectively, where r a parameter and

$$e_q^x = 1 + \frac{x}{[1]_q!} + \frac{x^2}{[2]_q!} + \dots + \frac{x^k}{[k]_q!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} \quad (7)$$

is q -analogue of the exponential function e^x and

$$[k]_q! = [1]_q [2]_q \dots [k]_q$$

is q -analogue of the factorial function $k! = 1.2.3 \dots k$. Thus, for q -Poisson Distribution, we have

$$P_q^r(x = k) = \frac{r^k}{[k]_q!} e_q^{-r}, \quad k = 1, 2, 3, \dots$$

Now, we introduce a power series whose coefficients are probabilities of the q -Poisson Distribution, that is

$$P_q^r(z) = z + \sum_{k=2}^{\infty} \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} z^k \quad (z \in U).$$

Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity. Now, for $r, s > 0$ and $0 < q < 1$, we introduce the analytic functions

$$P_q^r(z) = z + \sum_{k=2}^{\infty} \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} z^k \quad \text{and} \quad P_q^s(z) = z + \sum_{k=2}^{\infty} \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} z^k. \quad (8)$$

Let us define harmonic functions $P_q^{r,s}$ and $T^m P_q^{r,s}$ as

$$P_q^{r,s}(z) = P_q^r(z) + \overline{P_q^s(z)} - z \quad \text{and} \quad T^m P_q^{r,s} = 2z - P_q^r(z) + (-1)^{m-1} \overline{P_q^s(z)} - z. \quad (9)$$

It is clear that $P_q^{r,s} \in H^0$ and $T^m P_q^{r,s} \in T^{m,0}$.

In this study, we define two functions $P_q^{r,s}$ and $T^m P_q^{r,s}$ by q -Poisson Distribution and we aim to find the conditions for these functions to belong to the classes of q -starlike and q -convex harmonic functions.

2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

Lemma 1. [10] Let $f = h + \overline{g}$ be given by (1). If for some α ($0 \leq \alpha < 1$) and the inequality

$$\sum_{k=2}^{\infty} ([k]_q - \alpha) |a_k| + \sum_{k=1}^{\infty} ([k]_q + \alpha) |b_k| \leq 1 - \alpha \quad (10)$$

is hold, then f is harmonic, sense-preserving, univalent in U and $f \in SH_q^*(\alpha)$.

Remark 2. [10] Let $f = h + \overline{g}$ be given by (6). Then $f \in TSH_q^*(\alpha)$ if and only if the coefficient condition (10) is satisfied.

Lemma 3. [10] Let $f = h + \overline{g}$ be given by (1). If for some α ($0 \leq \alpha < 1$) and the inequality

$$\begin{aligned} & \sum_{k=2}^{\infty} [k]_q ([k]_q - \alpha) |a_k| \\ & + \sum_{k=1}^{\infty} [k]_q ([k]_q + \alpha) |b_k| \\ & \leq 1 - \alpha \end{aligned} \quad (11)$$

is hold, then f is harmonic, sense-preserving, univalent in U and $f \in KH_q(\alpha)$.

Remark 4. [10] Let $f = h + \bar{g}$ be given by (6). Then $f \in TKH_q(\alpha)$ if and only if the coefficient condition (11) is holds.

3. Main Results

Theorem 5. If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$ and the inequality

$$\begin{aligned} & r + s + e_q^{-r(1-q)} + e_q^{-s(1-q)} \\ & \leq (1 - \alpha)(1 + e_q^{-r}) + (1 + \alpha)e_q^{-s} \end{aligned} \quad (12)$$

is satisfied then $P_q^{r,s} \in SH_q^{*,0}(\alpha)$.

Proof. Let $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$. Referring Lemma 1, it is sufficient to show that the inequality

$$\begin{aligned} & \sum_{k=2}^{\infty} \left\{ ([k]_q - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \right\} + \\ & \sum_{k=2}^{\infty} \left\{ ([k]_q + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \right\} \leq 1 - \alpha \end{aligned} \quad (13)$$

is satisfied to show that the function $P_{p,q}^{r,s}(z) = P_q^r(z) + \overline{P_q^s(z)} - z$ belongs to the class $SH_q^{*,0}(\alpha)$ where P_q^r and P_q^s are given by (8). Then, using the inequality (7), we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \left\{ ([k]_q - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \right. \\ & \left. + ([k]_q + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \right\} \end{aligned}$$

$$\begin{aligned} & = \sum_{k=2}^{\infty} \left\{ ([k-1]_q + q^{k-1} - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \right. \\ & \left. + ([k-1]_q + q^{k-1} \right. \\ & \left. + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \right\} \end{aligned}$$

$$\begin{aligned} & = \sum_{k=2}^{\infty} \frac{r^{k-1} e_q^{-r}}{[k-2]_q!} + \sum_{k=2}^{\infty} (q^{k-1} - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \\ & + \sum_{k=2}^{\infty} \frac{s^{k-1} e_q^{-s}}{[k-2]_q!} \\ & + \sum_{k=2}^{\infty} (q^{k-1} + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \end{aligned}$$

$$\begin{aligned} & = r e_q^{-r} \sum_{k=0}^{\infty} \frac{r^k}{[k]_q!} + e_q^{-r} \sum_{k=1}^{\infty} \frac{(qr)^k}{[k]_q!} \\ & - \alpha e_q^{-r} \sum_{k=1}^{\infty} \frac{r^k}{[k]_q!} \\ & + s e_q^{-s} \sum_{k=0}^{\infty} \frac{s^k}{[k]_q!} + e_q^{-s} \sum_{k=1}^{\infty} \frac{(qs)^k}{[k]_q!} \\ & + \alpha e_q^{-s} \sum_{k=1}^{\infty} \frac{s^k}{[k]_q!} \end{aligned}$$

$$\begin{aligned} & = r + e_q^{-r} (e_q^{rq} - 1) - \alpha e_q^{-r} (e_q^r - 1) + s \\ & + e_q^{-s} (e_q^{sq} - 1) + \alpha e_q^{-s} (e_q^s - 1) \end{aligned}$$

Therefore, inequality (13) holds true if

$$\begin{aligned} & r + e_q^{-r} (e_q^{rq} - 1) - \alpha e_q^{-r} (e_q^r - 1) + s \\ & + e_q^{-s} (e_q^{sq} - 1) + \alpha e_q^{-s} (e_q^s - 1) \\ & \leq 1 - \alpha \end{aligned}$$

which is equivalent to (12). Thus, the proof of Theorem 5.

Corollary 6. If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$, then the function $T^1 P_q^{r,s}$ defined by (9) belongs to the class $TSH_q^{*,0}(\alpha)$ if and only if satisfied inequality (12).

Theorem 7. If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$ and the inequality

$$\begin{aligned} r^2 + s^2 + \alpha(s - r) + e_q^{-r(1-q^2)} + e_q^{-s(1-q^2)} \\ + [r(1 + 2q) - \alpha]e_q^{-r(1-q)} \\ + [s(1 + 2q) + \alpha]e_q^{-s(1-q)} \\ \leq (1 - \alpha)(1 + e_q^{-r}) \\ + (1 + \alpha)e_q^{-s} \end{aligned} \quad (14)$$

is satisfied then $P_q^{r,s} \in KH_q^0(\alpha)$.

Proof. Let $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$. Referring Lemma 2, it is sufficient to show that the inequality

$$\begin{aligned} \sum_{k=2}^{\infty} [k]_q \left\{ ([k]_q - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \right. \\ \left. + ([k]_q + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \right\} \\ \leq 1 - \alpha. \end{aligned} \quad (15)$$

is satisfied to show that the function $P_{p,q}^{r,s}(z) = P_q^r(z) + \overline{P_q^s(z)} - z$ belongs to the class $KH_q^0(\alpha)$ where P_q^r and P_q^s are given by (8). Then, using the inequality (7), we obtain

$$\begin{aligned} \sum_{k=2}^{\infty} [k]_q \left\{ ([k]_q - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \right. \\ \left. + ([k]_q + \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \right\} \\ = \sum_{k=2}^{\infty} \{ ([k-1]_q [k-2]_q \\ + (2q^{k-1} + q^{k-2} \\ - \alpha)[k-1]_q) \} \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \\ + \sum_{k=2}^{\infty} q^{k-1} (q^{k-1} - \alpha) \frac{r^{k-1} e_q^{-r}}{[k-1]_q!} \end{aligned}$$

$$\begin{aligned} + \sum_{k=2}^{\infty} \{ ([k-1]_q [k-2]_q \\ + (2q^{k-1} + q^{k-2} \\ + \alpha)[k-1]_q) \} \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \end{aligned}$$

$$\begin{aligned} + \sum_{k=2}^{\infty} q^{k-1} (q^{k-1} - \alpha) \frac{s^{k-1} e_q^{-s}}{[k-1]_q!} \\ = r^2 e_q^{-r} \sum_{k=0}^{\infty} \frac{r^k}{[k]_q!} + r(1 + 2q) e_q^{-r} \sum_{k=0}^{\infty} \frac{(qr)^k}{[k]_q!} \\ - \alpha r e_q^{-r} \sum_{k=0}^{\infty} \frac{r^k}{[k]_q!} + e_q^{-r} \sum_{k=1}^{\infty} \frac{(q^2 r)^k}{[k]_q!} \\ - \alpha e_q^{-r} \sum_{k=1}^{\infty} \frac{(qr)^k}{[k]_q!} + s^2 e_q^{-s} \sum_{k=0}^{\infty} \frac{s^k}{[k]_q!} \\ + s(1 + 2q) e_q^{-s} \sum_{k=0}^{\infty} \frac{(qs)^k}{[k]_q!} + \alpha s e_q^{-s} \sum_{k=0}^{\infty} \frac{s^k}{[k]_q!} \\ + e_q^{-s} \sum_{k=1}^{\infty} \frac{(q^2 s)^k}{[k]_q!} + \alpha e_q^{-s} \sum_{k=1}^{\infty} \frac{(qs)^k}{[k]_q!} \end{aligned}$$

$$\begin{aligned} = r^2 + r(1 + 2q) e_q^{-r(1-q)} - \alpha r + e_q^{-r(1-q^2)} \\ - e_q^{-r} - \alpha e_q^{-r(1-q)} + \alpha e_q^{-r} + s^2 \\ + s(1 + 2q) e_q^{-s(1-q)} + \alpha s \\ + e_q^{-s(1-q^2)} - e_q^{-s} + \alpha e_q^{-s(1-q)} \\ - \alpha e_q^{-s}. \end{aligned}$$

Therefore, inequality (15) holds true if

$$\begin{aligned} r^2 + r(1 + 2q) e_q^{-r(1-q)} - \alpha r + e_q^{-r(1-q^2)} \\ - e_q^{-r} - \alpha e_q^{-r(1-q)} + \alpha e_q^{-r} + s^2 \\ + s(1 + 2q) e_q^{-s(1-q)} + \alpha s \\ + e_q^{-s(1-q^2)} - e_q^{-s} + \alpha e_q^{-s(1-q)} \\ - \alpha e_q^{-s} \leq 1 - \alpha \end{aligned}$$

which is equivalent to (14). Thus, the proof of Theorem 7.

Corollary 8. If $r, s > 0, 0 < q < 1, 0 \leq \alpha < 1$, then the function $T^2 P_q^{r,s}$ defined by (9) belongs to the class $KH_q^0(\alpha)$ if and only if satisfied inequality (14).

4. Conclusion

The novelty of the above results consists in the fact that using some recent results we found sufficient conditions such that the function $P_q^{r,s}$ defined by (9) belongs to the classes $SH_q^{*,0}(\alpha)$ and $KH_q^0(\alpha)$ respectively.

Moreover, for appropriate choices of the parameters we found a few interesting special cases of the above main results.

Finally, new subclass analysis can be done using this method in the future.

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