A new subclass of meromorphic univalent functions associated with linear operator Lk involving complex order I

Zainab AlSaadawi[a] , Rafid H. Buti[b*]
[a] *School of Mathematics and Statistic , Central South University ,Changsha 410083, China .*

[b] *Department of Mathematics and Computer Applications, College of Science, AL-Muthanna University, Iraq.*
**Corresponding Author email: Sci.rafid@mu.edu.iq*

Received 2June 2022, accepted 7September 2022, published 31December 2022.

DOI: 10.52113/2/09.02.2022/16-21

Abstract: In this paper, we introduce a new subclass of analytic and meromorphic univalent function associated with liner operator involving complex order in the punctured unit disk. Then we characterized these functions and obtained some of properties of this subclass.

Keywords: Meromorphic function, Liner operator, univalent function, complex order.

2000 Mathematics Subject Classification.30C45.

1. Introduction

There are many researchers who have studied different classes of meromorphic univalent functions and meromorphic p −valent functions involving integral operators like Aouf [1], Atshan[2], Atshan and Kulkarni [3] , Ponnusamy [8], Tehranchi and Kulkarni [10] and this operator was studied by Urolgaddi and Somanatha [11]. Let RZ denote of the class of all meromorphic functions $f(z)$ defined as the following :

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (1)$$

Which are analytic and univalent in the punchred unit disk.

$$L^1\big(f(z)\big) = \frac{1}{z} \qquad a_n z_n + \sum_{n=1}^{\infty} (n+z) a_n z^n = \frac{z^2 f(z)}{z},$$

$$U^* = \{z : z \in \mathbb{C} \; and \; 0 < |z| < 1\} = U/\{0\}.$$

Let RZH denote the subclass of RZ consist of the functions defined as:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \, . \; a_n \geq 0. \qquad (2)$$

we define the Hadamared product (or convolution) of $f(z)$ and $g(z)$ by the form

$$(f \, * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

following the recent work of Liu and Sairastava [7] for a function belong to the class RZ given by (1) the Linear operator $L^k$is defind by:

$$L^0\big(f(z)\big) = f(z) \, ,$$

$$L^k f(z) = L\,[L^{k-1}f(z)] = \frac{1}{z} + \sum_{n=1}^{\infty}(n+2)^k a_n z^n = \frac{(z^2L^{k-1}f(z))'}{z}\ .k \in N\ . \quad (3)$$

It is easily verified from (4), that for $f \in RZ\ .k \in N$,

$$z[L^k f(z)]' = L^{k+1} f(z) - 2L^k f(z). \quad (5)$$

A function $f(z)$ defined by (2) belongs to the class RZH is said to be in the class RZH $(\alpha.m.\lambda.t.k)$ if and only if satisfies the following condition:

$$\left|\frac{\alpha[L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' + z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1)z^{-4}}{(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' - (1-m)z^{-3}[L^k f(z)] + (1-\alpha-\lambda)z^{-4}}\right| < 1 \quad (6)$$

Where $0 < \alpha \leq 1\,.0 \leq m < 1.\ 0 \leq \lambda < 1.t \in \mathbb{C}/\{0\}.\ k \in N.$

2. Characterization of the functions

We now investigate the coefficient characterization theorem for the function

$f(z) \in RZH(\alpha.m.\lambda.t.k$, then by obtaining the coefficient boundes .

Theorem 1 A function f(z) defined by (2) is in the class $RZH\ (\alpha.m.\lambda.t.k)$ if and only if

$\sum_{n=1}^{\infty}[\ [n(n-1)[\alpha(n-2) + 1 - m] + 1] + [\alpha n + 1 - m]]\ (n+2)^k an \leq (1-\lambda)|t|.$ (7)

Proof Let (7) holds true , for $|z| = 1$, we have

$$|\alpha[L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' + z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1]z^{-4}|$$

$$-|(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' - (1-m)z^{-3}[L^k f(z)] + (1-\alpha-m)z^{-4}|$$

$$= \left|\alpha[-6z^{-4} + \sum_{n=1}^{\infty} n(n-1)(n-2)(n+2)^k a_n z^{n-3}] + (1-m)z^{-1}[2z^{-3} + \sum_{n=1}^{\infty} n(n-1)(n+2)^k a_n z^{n-2}] + z^{-3}[z^{-1} + \sum_{n=1}^{\infty}(n-2)^k a_n z^n] - [2(1-m) + 6 \propto +1]z^{-4}\right|$$

$$-\left|(1-\lambda)t - \alpha z^{-2}[-z^{-2} + \sum_{n=1}^{\infty} n(n+2)^k a_n z^{n-1}] - (1-m)z^{-3}[z^{-1} + \sum_{n=1}^{\infty}(n+2)^k a_n z^n] + (1-\alpha)z^{-4}\right|$$

$= |-6\alpha z^{-4} + \sum_{n=1}^{\infty} \alpha\, n(n-1)(n-2)(n+2)^k a_n z^{n-3} + 2(1-m)z^{-4} + \sum_{n=1}^{\infty}(1-m)n(n-1)(n+2)^k a_n z^{n-3} + z^{-4} + \sum_{n=1}^{\infty} n+2)^k a_n z^{n-3} - [2(1-m) + 6\alpha + 1] - (1-\lambda)t + \alpha z^{-4} - \sum_{n=1}^{\infty} \alpha\, n(n+2)^k a_n z^{n-3} - (1-m)z^{-4} - \sum_{n=1}^{\infty}(1-m)(n+2)^k a_n z^{n-3} + (1-\alpha-m)z^{-4}|$

$= \quad |\sum_{n=1}^{\infty} \alpha\, n(n-1)(n-2)(n+2)^k a_n z^{n-3} + \sum_{n=1}^{\infty}(1-m)n(n-1)(n+2)a^k t^{n-3} + \sum_{n=1}^{\infty}(n+2)^k a_n z^{n-3}|$ - $|(1-\lambda)t - \sum_{n=1}^{\infty} \alpha\, n(n+2)^k a_n z^{n-3} - \sum_{n=1}^{\infty}(1-m)(n+2)^k a_n t^{n-3}|$

$= \quad |\sum_{n=1}^{\infty}[\alpha n(n-1)(n-2) + (1-m)n(n-1)+](n+2)^k a_n z^{n-3}||(1-\lambda)t - \sum_{n=1}^{\infty}[\propto n + (1-m)](n+2)^k a_n z^{n-3}|$

$= \quad |\sum_{n=1}^{\infty}[n(n-1)[\alpha(n-2)+(1-m)] + 1](n+2)^k a_n z^{n-3}| - |(1-\lambda)t - \sum_{n=1}^{\infty}[\alpha n + 1 - m] - (n+2)^k a_n z^{n-3}| \le \sum_{n=1}^{\infty}[n(n-1)[\propto (n-2) + 1 - m)] + 1](n+2)^k a_n |z|^{n-3} - (1-\lambda)|t| + \sum_{n=1}^{\infty}[\alpha n + 1 - m](n+2)^k a_n |z|^{n-3}$

For $|z|$=1 , me have

$= \quad \sum_{n=1}^{\infty}[(n(n-1)[\alpha(n-2)+1-m)] + 1] + [\alpha n + 1 - m]](n+2)^k a_n - (1-\lambda)|t|$

So , by hypothesis (7), we have $f(z)$ belongs to the class $RZH\ (\alpha.m.\lambda.t.k)$.

Conversely , assume that $f(z)$ is defined by (2) belongs to the class $RZH\ (\alpha.m.\lambda.t.k)$, from (6) , we have

$$\left| \frac{\alpha[L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' + z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1)z^{-4}}{(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' - (1-m)\, z^{-3}[L^k f(z)] + (1-\alpha-\lambda)z^{-4}} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty}[n(n-1)[\alpha(n-2)+(1-m)]+1](n+2)^k a_n z^{n-3}}{(1-\lambda)t - \sum_{n=1}^{\infty}[\alpha n + 1 - m](n+2)^k a_n z^{n-3}} \right|$$

Since $(z) \le |z|$, therefor ,we have

$$Re\left\{ \frac{\sum_{n=1}^{\infty} [n(n-1)[\alpha(n-2)+(1-m)] + 1](n+2)^k a_n z^{n-3}}{(1-\lambda)|t| - \sum_{n=1}^{\infty} [\alpha n + 1 - m](n+2)^k a_n z^{n-3}} \right\} < 1. \qquad (8)$$

Now, letting $z \to 1$, thought real values in (8) , at once obtain (7) and theorem is completely proved .

In the next theorem, we concentrated on getting the growth and distortion theorem for the $f(z)$ to belong in the class $(\alpha.m.\lambda.t.k)$.

Theoreme2 Let $f(z)$ belongs to the class $RZH\ (\alpha.m.\lambda.t.k)$

$$\frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k} \le |f(z)| \le \frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k} \quad . \quad (9)$$

This for $0 < |z| < 1$

Proof Let $f \in RZH\ (\alpha.m.\lambda.t.k)$. then

$$|f(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right| \le \left|\frac{1}{z}\right| + \sum_{n=1}^{\infty} a_n |z|$$

$$\le \left|\frac{1}{z}\right| + |z| \sum_{n=1}^{\infty} a_n$$

So, by using Theorem 1 , we have

$$\sum_{n=1}^{\infty} a_n \le \frac{(1-\lambda)|t|}{(2+\alpha-m)3^k},$$

thus

$$|f(z)| \le \frac{1}{|z|} + \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k}.$$

Similarly , we have

$$|f(z)| \ge \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n\ |z|^n \ge \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n\ .$$

thus

$$|f(z)| \ge \frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k}$$

Theorem 3

Let $f \in RZH(\alpha.m.\lambda.t.k)$. Then

$$\frac{1}{|z|^2}+\frac{(1-\lambda)|t|}{(2-m)3^k} \le |f'(z)| \le \frac{1}{|z|^2}+\frac{(1-\lambda)|t|}{(2-m)3^k}.$$

Proof Let $f \in RZH(\alpha.m.\lambda.t.k)$ we have $|f'(z)| \le \frac{1}{|z|^2}+\sum_{n=1}^{\infty} n\,a_n$, by using theorm 1 we have

$$\sum_{n=1}^{\infty} n\,a_n \le \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

So

$$|f'(z)| \le \frac{1}{|z|^2}+\frac{(1-\lambda)|t|}{(2-m)3^k}.$$

Similarly we have

$$|f'(z)| \le \frac{1}{|z|^2}+\sum_{n=1}^{\infty} n\,a_n \ge \frac{1}{|z|^2}-\frac{(1-\lambda)|t|}{(2-m)3^k}.$$

Thus we complete the proof .

3. The inclusion relation

Next, we determine the inclusion relationship involving $(n.s)$–neighborhoods.

Following the earlier works on neighborhoods of analytic function by Goodman [5], Ruscheweyh [9] and Atshan and Kulkarni [4] , but for meromerphic function studied by Lin and Srivastava, we define the $(n.s)$ -neighborhoods of function $f \in RZH$, by

$$N_{n.\delta}(f)=\{g\epsilon RZH : g(z)=\frac{1}{z}+\sum_{n=1}^{\infty} b_n z^n , \quad (11)$$

and

$$\sum_{n=1}^{\infty}|a_n-b_n| \le \delta.0 \le \delta < 1\} . \quad (12)$$

Definition 1 A function $g \in RZH$ is said to be in the class $RZH(\alpha.m.\lambda.t.k.\delta)$ if there exists a function $f \in RZH(\alpha.m.\lambda.t.k)$ such that

$$\left|\frac{g(z)}{f(z)}-1\right| < 1-\sigma, \quad z \in U, 0 \le \sigma < 1. \quad (13)$$

Theorm 4 Let $f \in RZH(\alpha.m.\lambda.t.k)$ and

$$\sigma = 1 - \frac{\delta(2+\alpha-m)3^k}{(2+\alpha-m)3^k-(1-\lambda)(t)} . \quad (14)$$

Then

$N_{n.\delta}(f) \subset RZH(\alpha.m.\lambda.t.k)$.

Proof Let $\in N_{n.\delta}(f)$, then we have from (11) that

$$\sum_{n=1}^{\infty} n|a_n-b_n| \le \delta,$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty}|a_n-b_n| \le \delta, n \in N.$$

Also since $\in RZH(\alpha.m.\lambda.t.k)$, we have from theorem (1)

$$\sum_{n=1}^{\infty} a_n \le \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

so that

$$\left|\frac{g(z)}{f(z)}-1\right| = \left|\frac{\sum_{n=1}^{\infty}(a_n-b_n)z^n}{\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n}\right| \le \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty} a_n}$$

$$\leq \frac{\delta(2+\alpha-m)3^k}{(2+\alpha-m)3^k-(L-\lambda)|t|} = 1-\delta.$$

Thus , by Definition 1, $g \in RZH$ for δ, is given by (13) .

This complete the proof. In the next , we considered integral transform of functions in the class $RZH\ (\alpha.m.\lambda.t.k)$.

Theorem 5 Let the function f given by (2) be in the class $RZH\ (\alpha.m.\lambda.t.k)$, then the integral operator

$$f(z) = c\int_0^1 u^c f(uz)du\,.\ 0 < u \leq 1.0 < c < \infty. \qquad (15)$$

in the class $RZH\ (\alpha.m.\lambda.t.k)$

Proof Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ in the class $RZH\ (\alpha.m.\lambda.t.k)$.

Then

$$f(z) = c\int_0^1 u^c f(uz)du$$

$$= c\int_0^1 u^c f(uz)du$$

$$= c\int_0^1 [\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} a_n u^{n-c}]du$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{c}{n+c+1}\right)a_n z^n.$$

It is easy to show that

$$\sum_{n=1}^{\infty} \frac{c\big[[n(n-1)][\alpha(n-2)+(1-m)]+1\big] + [\alpha n+1-m)](n-2)^k}{(c+n+1)(1-\lambda)|t|} a_n \leq 1\,. \quad (16)$$

Since

$f \in RZH\ (\alpha.m.\lambda.t.k)$,

$$\sum_{n=1}^{\infty} \frac{\big[[n(n-1)][\alpha(n-2)+(1-m)]+1\big] + [\alpha n+1-m)](n-2)^k}{(1-\lambda)|t|} a_n \leq 1.$$

note that (15) is satisfied

$$\frac{c[[n(n-1)][\alpha(n-2)+(1-m)]+1 + [an+1-m)](n-2)^k}{(c+n+1)(1-\lambda)|t|} \leq \frac{[[n(n-1)][\alpha(n-2)+(1-m)]+1 + [an+1-m)](n-2)^k}{(1-\lambda)|t|}.$$

Since $\frac{c}{c+n+1} < 1$ for all $\in N$. Hence , we obtained the required result.

References

[1] Aouf, M.K., , Acortian, 1989, Subclass of mebemerphically starlike functions with positive cofficients , Rendicenti di Mathematica , 9, 225-235 .

[2] Atshan, W.G., 2008, Subclass of meromorphic functions with positive coefficients defined by Ruschemeyh derivative II, J. Surveys in Math Appl., 3, 67-77.

[3] Atshan, W.G., Kulkarni, S.R., 2007, Subclass of meromorphic functions with positive

coefficients defined by Ruschemeyh derivative, I, J. Rajastan Acaed. Phys. Sci. 6(2), 129-140 .

[4] Atshan, W.G., Kulkarni, S.R., 2009, Negihborhoods and parial sunms of subclass of k-uniformly convex functions and related class of k-starlike functions with negative coefficients based on integral operator, J. Southeast Asian Bulletin of Math. 33(4), 623-637.

[5] Goodman, A.W., 1975, Univalent functions and non-analytic curve, Proc. Amer. Math. Soc. 8, 598-601.

[6] Liu, J. L., Srivastara, H.M ., 2001 A liner oprator and associated families of meromerphically multivalent functions, J. Math. Anal . Appl. 259, 566-581 .

[7] Liu, J. L., Srivastara, H.M., 2004, Subclass meromerphically multivalent functions associated with a certain liner oprator, Math. Comput. Modelling, 39, 35-44 .

[8] Ponnusamy, S., 1993, Convolution properties of some classes of meromerphic univalent functions , Proc. Indian Acad. Sci. Math. Sci., 103, 73-89.

[9] Ruscheweyh, S., 1981, neighborhoods of univalent functions, Proc. Amer. Math. Soc., 81 , 521-527.

[10] Tehranchi, A., Kulkarni, S.R., 2008, An application of differential subordination for the class of meromorphic multivalent functions with complex order, J. Southeast Asian Bulletin of Math., 32, 379-392.

[11] Uralegaddi, B.A., Somanatha, C.,1991 , New criteria for meromrphic starlike univalent functions, Bull. Austral. Math. Soc. 43, 137-140 .