A New Complex Double Integral Transform and It's Applications in Partial Differential Equations

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Abstract: In this paper, we present a new complex double integral transform namely "Complex Double Sadik Transform", to solve general linear partial differential equations. Several functions are used (applied) to show the usefulness of this new double transformation.

***Keywords*:** Complex Double Sadik Transform, Double Integral, Partial differential Equation, Sadik Integral Transform.

1. Introduction

Partial differential equations are critical in mathematical physics, The heat, wave, Laplace, Poisson, telegraphy and Kelin-Gordan equations are examples of fundamental equations in mathematical physics, that occur in a variety of branches of physcis, including applied mathematics and engineering, [1] .

Lots of researchers have presented many papers in double integral transform and their applications in partial differential equations and their applications in engineering, physics and other fields of life,[1-4],[7].

The researchers Emad A. and Eman A. Mansour presented in 2022 double SEE integral transform,[3]. In addition to, the researchers Saed M. Turq and Emad A. Kuffi presented in 2022 double of the Emad-Falih transformation and its properties with some important applications,[7].

The aim of this paper is to find the solution of linear partial differential equations by using (applying) a new complex double integral transformation (Complex Double Sadik transform).

Definition 1.1. [5] Sadik transform:

The Sadik integral transform of $g(t)$ is defined as:

$$\mathbf{S}_a[g(t)] = \mathbf{F}(v^\alpha, \beta) = \frac{1}{v^\beta}\int_0^\infty g(t)e^{-v^\alpha t}dt$$

Where v is a complex variable, α is any non zero real number, and β is any real number.

Definition 1.2. [6] The Complex Sadik Transform (CST) denoted by the operator $\mathbf{S}_a^c\{.\}$, the transform form is as follows:

$$\mathbf{S}_a^c[g(t)] = \mathbf{F}^c(s^\alpha, \beta) = \frac{1}{s^\beta}\int_0^\infty g(t)e^{-is^\alpha t}dt$$

Where s is a complex variable, α is a any nonzero real number, and β is any real number.

Definition 1.3. A New Complex Double Integral Transform denoted by the operator $\mathbf{D}^c\{.\}$, the transform form is as follows:

$$\mathbf{D}^c[f(x,t)] = \mathbf{F}^c(u,v)$$
$$= \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty f(x,t)e^{-i(u^\alpha x + v^\alpha t)}dtdx.$$

Where u, v are a complex variables, α is any non zero real number, and β is any real number

2. A New Complex Double Integral Transform Properties

In this Section, we introduce the new Complex Double Integral Transform of some famous functions:

1 Let $f(x,t) = 1$, then

$$\begin{aligned}\mathbf{D}^c[1] &= \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty 1e^{-i(u^\alpha x+v^\alpha t)}dtdx \\ &= \frac{1}{u^\beta}\int_0^\infty 1e^{-iu^\alpha x}dx \cdot \frac{1}{v^\beta}\int_0^\infty 1e^{-iv^\alpha t}dt \\ &= \mathbf{S}_a^c[1]\cdot \mathbf{S}_a^c[1] \\ &= \frac{-i}{u^{\alpha+\beta}}\cdot\frac{-i}{v^{\alpha+\beta}} \\ &= \frac{-1}{(uv)^{\alpha+\beta}}\end{aligned}$$

2 Let $f(x,t) = x^n t^m$, where n, m are positive integer numbers, then

$$\begin{aligned}\mathbf{D}^c[x^n t^m] &= \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty x^n t^m e^{-i(u^\alpha x+v^\alpha t)}dtdx \\ &= \frac{1}{u^\beta}\int_0^\infty x^n e^{-iu^\alpha x}dx \cdot \frac{1}{v^\beta}\int_0^\infty t^m e^{-iv^\alpha t}dt \\ &= \mathbf{S}_a^c[x^n]\cdot \mathbf{S}_a^c[t^n] \\ &= (-i)^{n+1}\frac{n!}{u^{n\alpha+(\alpha+\beta)}}\cdot(-i)^{m+1}\frac{m!}{v^{m\alpha+(\alpha+\beta)}} \\ &= \frac{(-i)^{n+m+2}n!\,m!}{u^{n\alpha+(\alpha+\beta)}\,.\,v^{m\alpha+(\alpha+\beta)}}\end{aligned}$$

3 Let $f(x,t) = e^{ax+bt}$, where a, b are real numbers,then

$$\begin{aligned}\mathbf{D}^c\left[e^{ax+bt}\right] &= \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty e^{ax+bt}e^{-i(u^\alpha x+v^\alpha t)}dtdx, \\ &= \frac{1}{u^\beta}\int_0^\infty e^{ax}e^{-iu^\alpha x}dx \cdot \frac{1}{v^\beta}\int_0^\infty e^{bt}e^{-iv^\alpha t}dt \\ &= \mathbf{S}_a^c[e^{ax}]\cdot \mathbf{S}_a^c\left[e^{bt}\right], \\ &= \frac{-1}{u^\beta}\left[\frac{a}{(u^{2\alpha}+a^2)} + i\frac{u^\alpha}{(u^{2\alpha}+a^2)}\right]\cdot\frac{-1}{v^\beta}\left[\frac{b}{(v^{2\alpha}+b^2)} + i\frac{v^\alpha}{(v^{2\alpha}+b^2)}\right], \\ &= \frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right].\end{aligned}$$

4 Let $f(x,t) = e^{-(ax+bt)}$, then

$$\begin{aligned}\mathbf{D}^c[e^{-(ax+bt)}] &= \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty e^{-(ax+bt)}e^{-i(u^\alpha x+v^\alpha t)}dtdx \\ &= \frac{1}{u^\beta}\int_0^\infty e^{-ax}e^{-iu^\alpha x}dx \cdot \frac{1}{v^\beta}\int_0^\infty e^{-bt}e^{-iv^\alpha t}dt \\ &= \mathbf{S}_a^c[e^{-ax}]\cdot \mathbf{S}_a^c[e^{-bt}] \\ &= \frac{-1}{u^\beta}\left[\frac{-a}{(u^{2\alpha}+(-a)^2)} + i\frac{u^\alpha}{(u^{2\alpha}+(-a)^2)}\right]\cdot\frac{-1}{v^\beta}\left[\frac{-b}{(v^{2\alpha}+(-b)^2)} + i\frac{v^\alpha}{(v^{2\alpha}+(-b)^2)}\right] \\ &= \frac{1}{(uv)^\beta}\left[\frac{(a-iu^\alpha)(b-iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\end{aligned}$$

5 Let $f(x,t) = e^{i(ax+bt)}$, then

$$D^c\left[e^{i(ax+bt)}\right] \quad \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty e^{i(ax+bt)}e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{u^\beta}\int_0^\infty e^{-ix(u^\alpha-a)}dx\cdot\frac{1}{v^\beta}\int_0^\infty e^{-it(v^\alpha-b)}dt$$
$$=\frac{1}{u^\beta}\frac{1}{-i(u^\alpha-a)}e^{-ix(u^\alpha-a)}\Big|_0^\infty\frac{1}{v^\beta}\frac{1}{-i(v^\alpha-b)}e^{-it(v^\alpha-b)}\Big|_0^\infty$$
$$=\frac{1}{u^\beta}\frac{1}{-i(u^\alpha-a)}[0-1]\frac{1}{v^\beta}\frac{1}{-i(v^\alpha-b)}[0-1]$$
$$=\frac{-i}{u^\beta(u^\alpha-a)}\frac{-i}{v^\beta(v^\alpha-b)}$$
$$=\frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}$$

6 Let $f(x,t)=e^{-i(ax+bt)}$, then

$$D^c\left[e^{-i(ax+bt)}\right]=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty e^{-i(ax+bt)}e^{-i(u^\alpha x+v^\alpha t)}dtdx,$$
$$=\frac{1}{u^\beta}\int_0^\infty e^{-ix(u^\alpha+a)}dx\cdot\frac{1}{v^\beta}\int_0^\infty e^{-it(v^\alpha+b)}dt,$$
$$=\frac{1}{u^\beta}\frac{1}{-i(u^\alpha+a)}e^{-ix(u^\alpha+a)}\Big|_0^\infty\frac{1}{v^\beta}\frac{1}{-i(v^\alpha+b)}e^{-it(v^\alpha+b)}\Big|_0^\infty,$$
$$=\frac{1}{u^\beta}\frac{1}{-i(u^\alpha+a)}[0-1]\frac{1}{v^\beta}\frac{1}{-i(v^\alpha+b)}[0-1],$$
$$=\frac{-i}{u^\beta(u^\alpha+a)}\frac{-i}{v^\beta(v^\alpha+b)},$$
$$=\frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}.$$

7 Let $f(x,t)=\sin(ax+bt)$, then

$$D^c[\sin(ax+bt)]=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \sin(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
, we know $\sin(ax+bt)=\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i}$
$$=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{2i}\left[D^c\left[e^{i(ax+bt)}\right]-D^c\left[e^{-i(ax+bt)}\right]\right],$$
$$=\frac{1}{2i}\left[\frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}+\frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}\right]$$
$$=\frac{1}{2i}\left[\frac{-1(u^\alpha+a)(v^\alpha+b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}+\frac{-1(u^\alpha-a)(v^\alpha-b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]$$
$$=i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]$$

8 Let $f(x,t)=\cos(ax+bt)$, then

$$D^c[\cos(ax+bt)]=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \cos(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
, we know $\cos(ax+bt)=\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2}$
$$=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{2}\left[D^c\left[e^{i(ax+bt)}\right]+D^c\left[e^{-i(ax+bt)}\right]\right]$$
$$=\frac{1}{2}\left[\frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}+\frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}\right]$$
$$=\frac{1}{2}\left[\frac{-1(u^\alpha+a)(v^\alpha+b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}+\frac{-1(u^\alpha-a)(v^\alpha-b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]$$
$$=\frac{-(u^\alpha v^\alpha+ab)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}$$

9 Let $f(x,t)=\sinh(ax+bt)$, then

$$D^c[\sinh(ax+bt)]=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \sinh(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{(ax+bt)}-e^{-(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{2}\left[D^c\left[e^{(ax+bt)}\right]-D^c\left[e^{-(ax+bt)}\right]\right]$$
$$=\frac{1}{2}\left[\frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]-\frac{1}{(uv)^\beta}\left[\frac{(a-iu^\alpha)(b-iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\right]$$
$$=i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]$$

10 Let $f(x,t)=\cosh(ax+bt)$, then

$$D^c[\cosh(ax+bt)]=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \cosh(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
, we know $\cosh(ax+bt)=\frac{e^{(ax+bt)}+e^{-(ax+bt)}}{2}$
$$=\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{(ax+bt)}+e^{-(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$=\frac{1}{2}\left[D^c\left[e^{(ax+bt)}\right]+D^c\left[e^{-(ax+bt)}\right]\right]$$
$$=\frac{1}{2}\left[\frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]+\frac{1}{(uv)^\beta}\left[\frac{(a-iu^\alpha)(b-iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\right]$$
$$=\frac{ab-u^\alpha v^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}$$

3. Summarization

The new complex double integral transform for some basic functions in the following table:

Table 1: The new complex double integral transform for some basic functions

Function	Complex Double Sadik transform
$f(x,t)$	$\mathbf{D}^c[f(x,t)]$
1	$\frac{-1}{(uv)^{\alpha+\beta}}$
$x^n t^m, m,n \in \mathbb{Z}^+$	$\frac{(-i)^{n+m+2} n!\, m!}{u^{n\alpha+(\alpha+\beta)} \cdot v^{m\alpha+(\alpha+\beta)}}$
e^{ax+bt}	$\frac{1}{(uv)^{\beta}}\left[\frac{(a+iu^{\alpha})(b+iv^{\alpha})}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]$
$\sin(ax+bt)$	$i\left[\frac{av^{\alpha}+bu^{\alpha}}{(uv)^{\beta}(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]$
$\cos(ax+bt)$	$\frac{-(u^{\alpha}v^{\alpha}+ab)}{(uv)^{\beta}(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}$
$\sinh(ax+bt)$	$i\left[\frac{av^{\alpha}+bu^{\alpha}}{(uv)^{\beta}(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]$
$\cosh(ax+bt)$	$\frac{ab-u^{\alpha}v^{\alpha}}{(uv)^{\beta}(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}$

4. Theorem and proof

$$\mathrm{D}^c\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx,$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}[\frac{1}{(u)^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iu^{\alpha}x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^{\alpha}x}$ $d\eta = \frac{\partial f(x,t)}{\partial x}dx$

$d\zeta = -iu^{\alpha}e^{-iu^{\alpha}x}dx$ $\eta = f(x,t)$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[\frac{1}{(u)^{\beta}}\left(\begin{array}{c} e^{-iu^{\alpha}x}f(x,t)\big|_0^{\infty} + \\ iu^{\alpha}\int_0^{\infty} f(x,t)e^{-iu^{\alpha}x}dx \end{array}\right)\right]dt$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[-\frac{f(0,t)}{u^{\beta}} + \frac{iu^{\alpha}}{u^{\beta}}\int_0^{\infty} f(x,t)e^{-iu^{\alpha}x}dx\right]dt$$

$$= -\frac{1}{u^{\beta}}\left[\frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}f(0,t)dt\right] + iu^{\alpha}\left[\frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} f(x,t)e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\right],$$

$$= -\frac{1}{u^{\beta}}\mathrm{F}^c(0,v) + iu^{\alpha}\mathrm{F}^c(u,v).$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial x^2}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx,$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}[\frac{1}{(u)^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial x^2}e^{-iu^{\alpha}x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^{\alpha}x}$ $d\eta = \frac{\partial^2 f(x,t)}{\partial x^2}dx$

$d\zeta = -iu^{\alpha}e^{-iu^{\alpha}x}dx$ $\eta = \frac{\partial f(x,t)}{\partial x}$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[\frac{1}{(u)^{\beta}}\left(e^{-iu^{\alpha}x}\frac{\partial f(x,t)}{\partial x}\Big|_0^{\infty} + iu^{\alpha}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iu^{\alpha}x}dx\right)\right]dt,$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[\frac{1}{(u)^{\beta}}e^{-iu^{\alpha}x}\frac{\partial f(x,t)}{\partial x}\Big|_0^{\infty} + \frac{iu^{\alpha}}{u^{\beta}}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iu^{\alpha}x}dx\right]dt,$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}[-\frac{1}{(u)^{\beta}}\frac{\partial f(0,t)}{\partial x} + \frac{iu^{\alpha}}{u^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iu^{\alpha}x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^\alpha x}\ d\eta = \dfrac{\partial f(x,t)}{\partial x}dx$

$d\zeta = -iu^\alpha e^{-iu^\alpha x}dx\ \eta = f(x,t)$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\Big[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial x}$$
$$+\frac{iu^\alpha}{u^\beta}\left(e^{-iu^\alpha x}f(x,t)\Big|_0^\infty + iu^\alpha\int_0^\infty f(x,t)e^{-iu^\alpha x}dx\right)\Big]dt$$
$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\Big[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial x}$$
$$+\frac{iu^\alpha}{u^\beta}\left(-f(0,t) + iu^\alpha\int_0^\infty f(x,t)e^{-iu^\alpha x}dx\right)\Big]dt,$$
$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\Big[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial x}$$
$$-\frac{iu^\alpha}{u^\beta}f(0,t) + \frac{(iu^\alpha)^2}{u^\beta}\int_0^\infty f(x,t)e^{-iu^\alpha x}dx\Big]dt,$$
$$= -\frac{1}{(u)^\beta}\left[\frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\frac{\partial f(0,t)}{\partial x}\right] - \frac{iu^\alpha}{u^\beta}\left[\frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}f(0,t)\right]$$
$$+(iu^\alpha)^2\left[\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty f(x,t)e^{-i(u^\alpha x+v^\alpha t)}dtdx\right],$$
$$= -\frac{1}{u^\beta}\frac{\partial \mathrm{F}^c(0,v)}{\partial x} - \frac{iu^\alpha}{u^\beta}\mathrm{F}^c(0,v) + (iu^\alpha)^2\mathrm{F}^c(u,v).$$

in general

$$let\ ab = \frac{\partial^{n-1}\mathrm{F}^c(0,v)}{\partial x^{n-1}} + iu^\alpha\frac{\partial^{n-2}\mathrm{F}^c(0,v)}{\partial x^{n-2}} + (iu^\alpha)^2\frac{\partial^{n-3}\mathrm{F}^c(0,v)}{\partial x^{n-3}}$$
$$\mathrm{D}^c\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^\alpha)^n\mathrm{F}^c(u,v) - \frac{1}{u^\beta}\left[\mathrm{ab} + \cdots + (iu^\alpha)^{n-2}\frac{\partial \mathrm{F}^c(0,v)}{\partial x} + (iu^\alpha)^{n-1}\mathrm{F}^c(0,v)\right].$$

or $\mathrm{D}^c\left[\dfrac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^\alpha)^n\mathrm{F}^c(u,v) - \dfrac{1}{u^\beta}\left[\sum_{k=1}^{n}(iu^\alpha)^{k-1}\dfrac{\partial^{n-k}\mathrm{F}^c(0,v)}{\partial x^{n-k}}\right]$

$$\mathrm{D}^c\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-i(u^\alpha x+v^\alpha t)}dtdx,$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[\frac{1}{(v)^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-iv^\alpha t}dt\Big]dx$$

Let $\zeta = e^{-iv^\alpha t}d\eta = \dfrac{df(x,t)}{dt}dt$

$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[\frac{1}{(v)^\beta}\left(e^{-iv^\alpha t}f(x,t)\Big|_0^\infty + iv^\alpha\int_0^\infty f(x,t)e^{-iv^\alpha t}dt\right)\right]dx$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[-\frac{f(x,0)}{v^\beta} + \frac{iv^\alpha}{v^\beta}\int_0^\infty f(x,t)e^{-iv^\alpha t}dt\right]dx$$
$$= -\frac{1}{v^\beta}\left[\frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}f(x,0)dx\right] + iv^\alpha\left[\frac{1}{(uv)^\beta}\int_0^\infty\int e_0^\infty f(x,t)e^{-i(u^\alpha x+v^\alpha t)}dtdx\right]$$
$$= -\frac{1}{v^\beta}\mathrm{F}^c(u,0) + iv^\alpha\mathrm{F}^c(u,v)$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial t^2}e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[\frac{1}{(v)^\beta}\underbrace{\int_0^\infty\frac{\partial^2 f(x,t)}{\partial t^2}e^{-iv^\alpha t}dt}_{\text{Integration by parts}}\Big]dx,$$

Let $\zeta = e^{-iv^\alpha t}\ d\eta = \dfrac{\partial^2 f(x,t)}{\partial t^2}dt$

$d\zeta = -iv^\alpha e^{-iv^\alpha t}dt\ \eta = \dfrac{\partial f(x,t)}{\partial t}$

$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[\frac{1}{(v)^\beta}\left(e^{-iv^\alpha t}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + iv^\alpha\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-iv^\alpha t}dt\right)\right]dx,$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[\frac{1}{(v)^\beta}e^{-iv^\alpha t}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + \frac{iv^\alpha}{v^\beta}\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-iv^\alpha t}dt\right]dx,$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[-\frac{1}{(v)^\beta}\frac{\partial f(x,0)}{\partial t} + \frac{iv^\alpha}{v^\beta}\underbrace{\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-iv^\alpha t}dt}_{\text{Integration by parts}}\Big]dx,$$

Let $\zeta = e^{-iv^\alpha t}\ d\eta = \dfrac{\partial f(x,t)}{\partial t}dt$

$d\zeta = -iv^\alpha e^{-iv^\alpha t}dt\ \eta = f(x,t)$

$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[-\frac{1}{(v)^\beta}\frac{\partial f(x,0)}{\partial t}$$
$$+\frac{iv^\alpha}{v^\beta}\left(e^{-iv^\alpha t}f(x,t)\Big|_0^\infty + iv^\alpha\int_0^\infty f(x,t)e^{-iv^\alpha t}dt\right)\Big]dx$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[-\frac{1}{(v)^\beta}\frac{\partial f(x,0)}{\partial t}$$
$$+\frac{iv^\alpha}{v^\beta}\left(-f(x,0) + iv^\alpha\int_0^\infty f(x,t)e^{-iv^\alpha t}dt\right)\Big]dx,$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[-\frac{1}{(v)^\beta}\frac{\partial f(x,0)}{\partial t}$$
$$-\frac{iv^\alpha}{v^\beta}f(x,0) + \frac{(iv^\alpha)^2}{v^\beta}\int_0^\infty f(x,t)e^{-iv^\alpha t}dt\Big]dx,$$
$$= -\frac{1}{(v)^\beta}\left[\frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\frac{\partial f(x,0)}{\partial t}\right] - \frac{iv^\alpha}{v^\beta}\left[\frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}f(x,0)\right]$$
$$+(iv^\alpha)^2\left[\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty f(x,t)e^{-i(u^\alpha x+v^\alpha t)}dtdx\right],$$
$$= -\frac{1}{v^\beta}\frac{\partial \mathrm{F}^c(u,0)}{\partial t} - \frac{iv^\alpha}{v^\beta}\mathrm{F}^c(u,0) + (iv^\alpha)^2\mathrm{F}^c(u,v).$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial t\partial x}\right] = \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial t\partial x}e^{-i(u^\alpha x+v^\alpha t)}dtdx$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\Big[\frac{1}{(v)^\beta}\underbrace{\int_0^\infty\frac{\partial^2 f(x,t)}{\partial t\partial x}e^{-iv^\alpha t}dt}_{\text{Integration by parts}}\Big]dx,$$

Let $\zeta = e^{-iv^\alpha t}\ d\eta = \dfrac{\partial^2 f(x,t)}{\partial t\partial x}dt$

$d\zeta = -iv^\alpha e^{-iv^\alpha t}dt\ \eta = \dfrac{\partial f(x,t)}{\partial x}$

$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[\frac{1}{(v)^\beta}\left(e^{-iv^\alpha t}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty + iv^\alpha\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-iv^\alpha t}dt\right)\right]dx,$$
$$= \frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\left[\frac{1}{(v)^\beta}e^{-iv^\alpha t}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty + \frac{iv^\alpha}{v^\beta}\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-iv^\alpha t}dt\right]dx,$$
$$= -\frac{1}{v^\beta}\left[\frac{1}{(u)^\beta}\int_0^\infty e^{-iu^\alpha x}\frac{\partial f(x,0)}{\partial x}\right] + iv^\alpha\left[\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-i(u^\alpha x+v^\alpha t)}dtdx\right],$$
$$= -\frac{1}{v^\beta}\frac{\partial \mathrm{F}^c(u,0)}{\partial x} + iv^\alpha\left[-\frac{1}{u^\beta}\mathrm{F}^c(0,v) + iu^\alpha\mathrm{F}^c(u,v)\right],$$
$$= -\frac{1}{v^\beta}\frac{\partial \mathrm{F}^c(u,0)}{\partial x} - \frac{iv^\alpha}{u^\beta}\mathrm{F}^c(0,v) + i^2u^\alpha v^\alpha\mathrm{F}^c(u,v)$$

In general

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\frac{\partial^{n-1}\mathbf{F}^c(u,0)}{\partial t^{n-1}} + iv^\alpha \frac{\partial^{n-2}\mathbf{F}^c(u,0)}{\partial t^{n-2}} + (iv^\alpha)^2 \frac{\partial^{n-3}\mathbf{F}^c(u,0)}{\partial t^{n-3}} + \cdots + (iv^\alpha)^{n-2}\frac{\partial \mathbf{F}^c(u,0)}{\partial t} + (iv^\alpha)^{n-1}\mathbf{F}^c(u,0)\right]$$

$$\text{or } = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^{n} (iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right]$$

$$\mathbf{D}^c\left[\frac{\partial^2 f(x,t)}{\partial x \partial t}\right] = \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty \frac{\partial^2 f(x,t)}{\partial x \partial t} e^{-i(u^\alpha x + v^\alpha t)} dt dx,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}[\frac{1}{(u)^\beta}\underbrace{\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x \partial t} e^{-iu^\alpha x} dx}_{\text{Integration by parts}}] dt,$$

Let $\zeta = e^{-iu^\alpha x}$ $d\eta = \frac{\partial^2 f(x,t)}{\partial x \partial t}dx$

$d\zeta = -iu^\alpha e^{-iu^\alpha x} dx$ $\eta = \frac{\partial f(x,t)}{\partial t}$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta}\left(e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + iu^\alpha \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right)\right] dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta} e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + \frac{iu^\alpha}{u^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right] dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial t} + \frac{iu^\alpha}{u^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right] dt$$

$$= -\frac{1}{u^\beta}\left[\frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\frac{\partial f(0,t)}{\partial t}\right] + iu^\alpha\left[\frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-i(u^\alpha x + v^\alpha t)} dt dx\right],$$

$$= -\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} + iu^\alpha\left[-\frac{1}{v^\beta}\mathbf{F}^c(u,0) + iv^\alpha \mathbf{F}^c(u,v)\right],$$

$$= -\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} - \frac{iu^\alpha}{v^\beta}\mathbf{F}^c(u,0) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v).$$

Tabel(2): Summarization

Function	Complex Double Sadik transform
$f(x,t)$	$\mathbf{D}^c[f(x,t)] = \mathbf{F}^c(u,v)$
$\frac{\partial f(x,t)}{\partial x}$	$-\frac{1}{u^\beta}\mathbf{F}^c(0,v) + iu^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x^2}$	$-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial x} - \frac{iu^\alpha}{u^\beta}\mathbf{F}^c(0,v) + (iu^\alpha)^2 \mathbf{F}^c(u,v)$
$\frac{\partial^n f(x,t)}{\partial x^n}$	$(iu^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{n}(iu^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(0,v)}{\partial x^{n-k}}\right]$
$\frac{\partial f(x,t)}{\partial t}$	$-\frac{1}{v^\beta}\mathbf{F}^c(u,0) + iv^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial t^2}$	$-\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial t} - \frac{iv^\alpha}{v^\beta}\mathbf{F}^c(u,0) + (iv^\alpha)^2 \mathbf{F}^c(u,v)$
$\frac{\partial^n f(x,t)}{\partial t^n}$	$(iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^{n}(iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right]$
$\frac{\partial^2 f(x,t)}{\partial t\,\partial x}$	$-\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial x} - \frac{iv^\alpha}{u^\beta}\mathbf{F}^c(0,v) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x\,\partial t}$	$-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} - \frac{iu^\alpha}{v^\beta}\mathbf{F}^c(u,0) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v)$

Theorem 4.1. Let $\mathbf{F}^c(u,v)$ is the complex new integral transform of $f(x,t)(\mathbf{F}^c(u,v) = \mathbf{D}^c[f(x,t)])$, then

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{n} (iu^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(0,v)}{\partial x^{n-k}}\right] \quad (1)$$

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^{n} (iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right] \quad (2)$$

Proof. Firstly, We want prove (1) by mathematical induction

1 for $n = 1$

$$\mathbf{D}^c\left[\frac{\partial f(x,t)}{\partial x}\right] = (iu^\alpha)^1 \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{1} (iu^\alpha)^{k-1}\frac{\partial^{1-k}\mathbf{F}^c(0,v)}{\partial x^{1-k}}\right]$$

$$= iu^\alpha \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\mathbf{F}^c(0,v)$$

Thus, true for $n = 1$

2 Assume true for $n = m$ that mean

$$\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial x^m}\right] = (iu^\alpha)^m \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{m} (iu^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}}\right]$$

3 we want to prove (1) for $n = m + 1$

$$\mathrm{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right]=\mathrm{D}^c\left[\frac{\partial}{\partial x}\left[\frac{\partial^m f(x,t)}{\partial x^m}\right]\right]$$

$$=iu^\alpha \mathrm{D}^c\left[\frac{\partial^m f(x,t)}{\partial x^m}\right]-\frac{1}{u^\beta}\frac{\partial^m \mathrm{F}^c(0,v)}{\partial x^m},$$

$$=iu^\alpha\left[(iu^\alpha)^m \mathrm{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k-1}\frac{\partial^{m-k}\mathrm{F}^c(0,v)}{\partial x^{m-k}}\right]\right]-\frac{1}{u^\beta}\frac{\partial^m \mathrm{F}^c(0,v)}{\partial x^m},$$

$$=(iu^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(0,v)}{\partial x^{m-k}}\right]-\frac{1}{u^\beta}\frac{\partial^m \mathrm{F}^c(0,v)}{\partial x^m},$$

$$=(iu^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(0,v)}{\partial x^{m-k}}+\frac{\partial^m \mathrm{F}^c(0,v)}{\partial x^m}\right],$$

$$=(iu^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=0}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(0,v)}{\partial x^{m-k}},$$

$$=(iu^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=1}^{m+1}(iu^\alpha)^{k-1}\frac{\partial^{m-(k-1)}\mathrm{F}^c(0,v)}{\partial x^{m-(k-1)}},$$

$$=(iu^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=1}^{m+1}(iu^\alpha)^{k-1}\frac{\partial^{m+1-k}\mathrm{F}^c(0,v)}{\partial x^{m+1-k}},$$

$$=\mathrm{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right]$$

So Theorem is true for $n \in \mathbb{N}$

Finally, We want prove (2) by mathematical induction

1 for $n = 1$

$$\mathbf{D}^c\left[\frac{\partial f(x,t)}{\partial t}\right] = (iv^\alpha)^1\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{1}(iv^\alpha)^{k-1}\frac{\partial^{1-k}\mathbf{F}^c(u,0)}{\partial t^{1-k}}\right]$$

$$= iv^\alpha\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\mathbf{F}^c(u,0)$$

Thus, true for $n = 1$

2 Assume true for $n = m$ that mean

$$\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial t^m}\right] = (iv^\alpha)^m\mathbf{F}^c(u,v) -\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}}\right]$$

3 We want to prove (2) for $n = m + 1$

$$\mathrm{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right]=\mathrm{D}^c\left[\frac{\partial}{\partial t}\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]\right]$$

$$=iv^\alpha \mathrm{D}^c\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]-\frac{1}{v^\beta}\frac{\partial^m \mathrm{F}^c(u,0)}{\partial t^m},$$

$$=iv^\alpha\left[(iv^\alpha)^m \mathrm{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k-1}\frac{\partial^{m-k}\mathrm{F}^c(u,0)}{\partial t^{m-k}}\right]\right]-\frac{1}{v^\beta}\frac{\partial^m \mathrm{F}^c(u,0)}{\partial t^m},$$

$$=(iv^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(u,0)}{\partial t^{m-k}}\right]-\frac{1}{v^\beta}\frac{\partial^m \mathrm{F}^c(u,0)}{\partial t^m},$$

$$=(iv^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(u,0)}{\partial t^{m-k}}+\frac{\partial^m \mathrm{F}^c(u,0)}{\partial t^m}\right],$$

$$=(iv^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=0}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathrm{F}^c(u,0)}{\partial t^{m-k}},$$

$$=(iv^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=1}^{m+1}(iv^\alpha)^{k-1}\frac{\partial^{m-(k-1)}\mathrm{F}^c(u,0)}{\partial t^{m-(k-1)}},$$

$$=(iv^\alpha)^{m+1}\mathrm{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=1}^{m+1}(iv^\alpha)^{k-1}\frac{\partial^{m+1-k}\mathrm{F}^c(u,0)}{\partial t^{m+1-k}},$$

$$=\mathrm{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right]$$

So Theorem is true for $n \in \mathbb{N}$

5. Example

Example 5.1. Consider the following partial differential equation

$$u_{tt} = u_{xx} \qquad (3)$$

with conditions

$$u(x,0) = \sin x, u_t(x,0) = 2$$
$$u(0,t) = 2t, u_x(0,t) = \cos t$$

Solution:

we take Double complex Sadik transform in Equation (3)

$$\mathbf{D}^c[u_{tt}] = \mathbf{D}^c[u_{xx}]$$

we obtain

$$-\frac{1}{v^{\beta}}\frac{\partial \mathrm{F}^{c}(u,0)}{\partial t}-\frac{iv^{\alpha}}{v^{\beta}}\mathrm{F}^{c}(u,0)+(iv^{\alpha})^{2}\mathrm{F}^{c}(u,v)$$
$$=-\frac{1}{u^{\beta}}\frac{\partial \mathrm{F}^{c}(0,v)}{\partial x}-\frac{iu^{\alpha}}{u^{\beta}}\mathrm{F}^{c}(0,v)$$
$$+(iu^{\alpha})^{2}\mathrm{F}^{c}(u,v)$$

$$-(iu^{\alpha})^{2}\mathbf{F}^{c}(u,v)+(iv^{\alpha})^{2}\mathbf{F}^{c}(u,v)=\frac{1}{v^{\beta}}\frac{\partial \mathbf{F}^{c}(u,0)}{\partial t}+\frac{iv^{\alpha}}{v^{\beta}}\mathbf{F}^{c}(u,0)-\frac{1}{u^{\beta}}\frac{\partial \mathbf{F}^{c}(0,v)}{\partial x}-\frac{iu^{\alpha}}{u^{\beta}}\mathbf{F}^{c}(0,v)$$
$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=\frac{1}{v^{\beta}}\frac{\partial \mathbf{F}^{c}(u,0)}{\partial t}+\frac{iv^{\alpha}}{v^{\beta}}\mathbf{F}^{c}(u,0)-\frac{1}{u^{\beta}}\frac{\partial \mathbf{F}^{c}(0,v)}{\partial x}-\frac{iu^{\alpha}}{u^{\beta}}\mathbf{F}^{c}(0,v)$$

substitute $\mathbf{F}^{c}(u,0),\mathbf{F}^{c}(0,v),\frac{\partial \mathbf{F}^{c}(0,v)}{\partial x}$ and $\frac{\partial \mathbf{F}^{c}(u,0)}{\partial t}$, we obtain

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=\frac{1}{v^{\beta}}\frac{-2i}{u^{\alpha+\beta}}+\frac{iv^{\alpha}}{v^{\beta}}\frac{-1}{u^{\beta}(u^{2\alpha}-1)}-\frac{1}{u^{\beta}}\frac{-iv^{\alpha}}{v^{\beta}(v^{2\alpha}-1)}-\frac{iu^{\alpha}}{u^{\beta}}\frac{-2}{v^{2\alpha+\beta}}$$
$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=-\frac{2i}{(uv)^{\beta}u^{\alpha}}-\frac{iv^{\alpha}}{(uv)^{\beta}(u^{2\alpha}-1)}+\frac{iv^{\alpha}}{(uv)^{\beta}(v^{2\alpha}-1)}+\frac{2iu^{\alpha}}{(uv)^{\beta}v^{2\alpha}}$$
$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=\frac{2iu^{\alpha}}{(uv)^{\beta}v^{2\alpha}}-\frac{2i}{(uv)^{\beta}u^{\alpha}}+\frac{iv^{\alpha}}{(uv)^{\beta}(v^{2\alpha}-1)}-\frac{iv^{\alpha}}{(uv)^{\beta}(u^{2\alpha}-1)}$$
$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=\frac{2i}{(uv)^{\beta}}\left[\frac{u^{\alpha}}{v^{2\alpha}}-\frac{1}{u^{\alpha}}\right]+\frac{iv^{\alpha}}{(uv)^{\beta}}\left[\frac{1}{v^{2\alpha}-1}-\frac{1}{u^{2\alpha}-1}\right]$$
$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^{c}(u,v)=\frac{2i(u^{2\alpha}-v^{2\alpha})}{(uv)^{\beta}v^{2\alpha}u^{\alpha}}+\frac{iv^{\alpha}(u^{2\alpha}-v^{2\alpha})}{(uv)^{\beta}(u^{2\alpha}-1)(v^{2\alpha}-1)}$$

Division by $u^{2\alpha}-v^{2\alpha}$, we get

$$\mathbf{F}^{c}(u,v)=\frac{2i}{(uv)^{\beta}v^{2\alpha}u^{\alpha}}$$
$$+\frac{iv^{\alpha}}{(uv)^{\beta}(u^{2\alpha}-1)(v^{2\alpha}-1)}$$

Take inverse of both side, we obtain

$$u(x,t)=2t+\sin x\cos t$$

Conclusion

In this work, a new Complex double integral transformation is introduced for finding the solution of general partial differential equations.

References

[1] Debnath, L., 2016, The double laplace transforms and their properties with applications to functional, integral and partial differential equations. Int. J. Appl. Comput. Math, 2, 223-241.

[2] Eltayeb, H., and Kiliçman, A., 2010. On double sumudu transform and double laplace transform. Malaysian Journal of Mathematical Sciences, 4(1), 17-30,

[3] Kuffi, E. A., and Mansour, E. A., 2022, Solving partial differential equations using the new integral transform "double see integral transform". Journal of physics: Conference Series, 2322(1), 1-5.

[4] Patil, D. P., 2020, Dualities between double integral transforms. International Advanced Research Journal in Science, Engineering and Technology IARJSET, 7(6), 17-30,

[5] Shaikh, S. L., 2018, Introducing a new integral transform: Sadik transform. American International Journal of Research in Science, Technology, Engineering Mathematics, 22(1), $100-102$.

[6] Turq, S. M., and Kuffi, E. A., 2022, The new complex integral transform 'complex sadik transform' and it's applications. Ibn Al-Haitham Journal for Pure and Applied sciences, 35(3).

[7] Turq, S. M., and Kuffi E. A., 2022, On the double of emad-falih transformation and its properties with applications. Ibn Al-Haitham Journal for Pure and Applied sciences, 35(4):220-234.