On a Class of Analytic Univalent Functions Associated with (H-R) Fractional Derivative

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ABSTRAC: In this paper, we introduced a class of analytic functions defined by (H–R) fractional derivative defined in the unit disk, we obtained coefficient bounded, so we obtained some theorems of this class. 2000Mathematics Subject classification: 30C45.

Keywords: univalent function, (H- R) fractional derivative, Hadamard product.

1. Introduction

Let *RHB* denoted the class of functions of the form:

$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0(n \in \{1,2,\dots\})$ …………. (1)

which are analytic and univalent function in the open unit disk:

$U = \{z \in C: |z| < 1\}.$

Given $g \in RHB$, f given by (1), and then the Hadamard product ((or convolution)) defined by:

$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)$ …………. (2)

Where:

$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, (b_n \geq 0, n \in \{1,2,\dots\})$ ………….(3)

In the next we defined ((H-R)) fractional derivative.

Defintion [1]: The fractional derivative of order δ□δ □ 2,3,...□ is defined by:

${}_z^{\delta}Df(z) = \frac{1}{\Gamma(\delta-1)} \int_0^z (z-u)^{\delta-2} f(u)du$

…………. (4)

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Where $f(z)$ is the analytic function in a simply connected region of z-plan, counting the origin, and the multiplicity □$z-u$□$^{\delta-2}$ is removed by required log□$z-u$□ to be real when □$z-u$□□ 0

From Definition 1 by being applying simple calculations, we obtain:

$G(z) = z^{2-\delta}\Gamma(\delta)\,{}_z^{\delta}Df(z) = z - \sum_{n=2}^{\infty} \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n$ ……….. (5)

Many authors have studied fractional calculus like, Atshan W. G. [2] Atshan W. G. and Kulkarni [3], S.R. Choi J.H., Kim Y.C. and Owa S. [4], Goyal S.D. and Goswami P. [5].

Definition 1: A function $f \in RHB$ is said to be in the class $RHB(\delta,\alpha,A,\beta,\lambda,\ell,\Omega)$ if and only if satisfies the next condition.

$$\left|\frac{z^2\ell\left(G''(z)\right)}{\alpha(1-A)-\lambda\beta z\left(G'(z)-1\right)}\right| < 1-\Omega ,$$

where $0 \le \alpha \le 1, \lambda \ge 0, \beta \ge 0, \ell > 0, \Omega < 1, z \in U, 0 \le A < 1$

2. Main Results

Theorem1: A function f be in the class $RHB(\delta,\alpha,A,\beta,\lambda,\ell,\Omega)$ if and only if

$$\sum_{n=2}^{\infty} n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(1-\lambda\beta)a_n \le \alpha(1-A) \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (6)$$

Proof: Assume that the inequality (6) holds true and let $|z|=1$,

$$\left|\frac{z^2\ell\left(G''(z)\right)}{\alpha(1-A)-\lambda\beta z\left(G'(z)-1\right)}\right| < 1$$

$$= \left|\frac{z^2\ell\left(\sum_{n=2}^{\infty} n(n-1)\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n z^{n-1}\right)}{\alpha(1-A)-\lambda\beta z\left(1-\sum_{n=2}^{\infty} n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n z^{n-1}-1\right)}\right|$$

$$= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)\ell\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n z^{n}}{\alpha(1-A)+\sum_{n=2}^{\infty}\lambda\beta n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n z^{n}}\right| < 1-\Omega$$

$$put \quad \Psi(n,\delta) = \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}$$

$$= \left|\sum_{n=2}^{\infty} n(n-1)\ell\Psi(n,\delta)a_n z^n\right| < (-\Omega)\left|\alpha(1-A)+\sum_{n=2}^{\infty}\lambda\beta n\Psi(n,\delta)a_n z^n\right|$$

$$= \left|\sum_{n=2}^{\infty} n(n-1)\ell\Psi(n,\delta)a_n z^n\right| - (1-\Omega)\left|\alpha(1-A)+\lambda\beta n\Psi(n,\delta)a_n z^n\right| < 0$$

$$= \sum_{n=2}^{\infty} n(n-1)\lambda\Psi(n,\delta)a_n|z|^n - (1-\Omega)|\alpha(1-A)| - \sum_{n=2}^{\infty}\lambda\beta n\Psi(n,\delta)\,a_n|z|^n$$

$$= \sum_{n=2}^{\infty} n(n-1)\ell\Psi(n,\delta)a_n - \alpha(1-\Omega)(1-A) - \sum_{n=2}^{\infty}\lambda\beta n\Psi(n,\delta)\,a_n$$

$$= \sum_{n=2}^{\infty} n(n-1)\ell\Psi(n,\delta)\,a_n(1-\lambda\beta) - \alpha(1-\Omega)(1-A) \le 0$$

$$= \sum_{n=2}^{\infty} n(n-1)\ell\Psi(n,\delta)(1-\lambda\beta)\,a_n \le \alpha(1-\Omega)(1-A)$$

Conversely, suppose that f is in the class $RHB(\delta,\alpha,A,\beta,\lambda,\ell,\Omega)$

$$\left|\frac{z^2\ell\left(\sum_{n=2}^{\infty} n(n-1)\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n z^{n-2}\right)}{\alpha(1-A)-\lambda\beta z\left(1-\sum_{n=2}^{\infty} n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}a_n\, z^{n-1}-1\right)}\right| < 1-\Omega$$

$$\left|\frac{\sum_{n=2}^{\infty} n(n-1)\ell \frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n}{\alpha(1-A) - \lambda\beta \sum_{n=2}^{\infty} n \frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n}\right| < 1 - \Omega$$

For all z, we have $|\mathrm{Re}(z)| \le |z|$ sine

$$Re\left\{\frac{\sum_{n=2}^{\infty} n(n-1)\ell \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n}{\alpha(1-A) - \sum_{n=2}^{\infty} \lambda\beta n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n z^n}\right\} < 1 - \Omega$$

Choose the value of z on the real axis and $z \to 1$, we obtain

$$\sum_{n=2}^{\infty} n(n-1)\ell \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n \le \alpha(1-\Omega)(1-A) + \sum_{n=2}^{\infty} \lambda\beta n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} a_n$$

So,

$$\sum_{n=2}^{\infty} n \frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta) a_n \le \alpha(1-\Omega)(1-A)$$

Corollary (2):

Let $f \in RHB(\delta, \alpha, A, \beta, \lambda, \ell, \Omega)$, then:

$$a_n \le \frac{\alpha(1-\Omega)(1-A)}{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)}.$$

Theorem (3): Let the function defined by (1) be in the class $RHB(\delta, \alpha, A, \beta, \lambda, \ell, \Omega)$, then $r - \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} r^2 \le |f(z)| \le r + \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} r^2$ ……… (7)

$0 < |z| < r < 1$

The equality in (7) is attained by the function f given by:

$$f(z) = z - \frac{\alpha(1-\Omega)(1-A)}{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)} z^2$$

Proof: since the function f defined by (1) in the $RHB(\delta, \alpha, A, \beta, \lambda, \ell, \Omega)$ we have from theorem(1).

, Thus

$$\sum_{n=2}^{\infty} a_n \le \frac{\alpha(1-\Omega)(1-A)}{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)}$$

$$|f(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \le |z| + \sum_{n=2}^{\infty} a_n |z|^n$$

$$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

$$|f(z)| \le r + \frac{\alpha(1-\Omega)(1-A)}{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)} r^2$$

Similarly,

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n$$

$$|f(z)| \ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$|f(z)| \ge r - r^2 \sum_{n=2}^{\infty} a_n$$

$$|f(z)| \ge r - \frac{\alpha(1-\Omega)(1-A)}{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)} r^2$$

This completes the proof.

Theorem (4):

Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n$$

Then f is in the class $RHB(\delta,\alpha,A,\beta,\lambda,\ell,\Omega)$ if and only if it can be expressed in the from $f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$

where:

$(\sigma_n f_n(z)$ and $\sum_{n=1}^{\infty} \sigma_n = 1$ or $1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n)$

Proof:

Suppose that $f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$

$f(z) = \sigma_1 f_1(z) + \sum_{n=2}^{\infty} \sigma_n f_n(z)$ ……………. (8)

$$f(z) = \sigma_1 f_1(z) - \sum_{n=2}^{\infty} \sigma_n \left[z - \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n \right]$$

$$f(z) = z\left(\sigma_1 + \sum_{n=2}^{\infty} \sigma_n\right) - \sum_{n=2}^{\infty} \sigma_n \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n$$

$$f(z) = z - \sum_{n=2}^{\infty} \sigma_n \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n$$

From theorem (1)

$$a_n \le \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)}$$

Setting

$$\sigma_n = \frac{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)}{\alpha(1-\Omega)(1-A)} a_n$$

$$a_n = \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} \sigma_n$$

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

$$f(z) = z - \sum_{n=2}^{\infty} \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} \sigma_n z^n$$

From

$$f_n(z) = z - \frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\,\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n$$

$$\frac{\alpha(1-\Omega)(1-A)}{n\frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)}(\ell(n-1)-\lambda\beta)} z^n = z - f_n(z)$$

Then

$$f(z) = z - \sum_{n=2}^{\infty} \sigma_n (z - f_n(z))$$

$$f(z) = z - \sum_{n=2}^{\infty} \sigma_n z + \sum_{n=2}^{\infty} \sigma_n f_n(z)$$

$$f(z) = z\left(1 - \sum_{n=2}^{\infty} \sigma_n\right) + \sum_{n=2}^{\infty} \sigma_n f_n(z)$$

$$f(z) = f_1\sigma_1 + \sum_{n=2}^{\infty} \sigma_n f_n(z)$$

$$f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$$

This completes the proof.

Theorem (5):

Let the function f defined by (1) be in the class $RHB(\delta, \alpha, \mathrm{A}, \beta, \lambda, \ell, \Omega)$ for every $(r = 2,3,\dots,m)$. Then the arithmetic mean of $f_r\ (r = 2,3,\dots,m)$ is defined by:

$$g(z) = z - \sum_{n=2}^{\infty} c_n z^n$$

$$(c_n \geq 2, n \geq 2, n \in N)$$

Also belongs to the class $RHB(\delta, \alpha, \mathrm{A}, \beta, \lambda, \ell, \Omega)$, where $c_n = \frac{1}{m}\sum_{r=2}^{\infty} a_{n,r}$

Proof: Since $f_r \in RHB(\delta, \alpha, \mathrm{A}, \beta, \lambda, \ell, \Omega)$, then from theorem (1), we get

$$\sum_{n=2}^{\infty} n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta) a_{n,r} \leq \alpha(1-\Omega)(1-A)$$

……………… (9)

$$\sum_{n=2}^{\infty} n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta) c_n$$

$$\sum_{n=2}^{\infty} n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta) \left[\frac{1}{m}\sum_{r=2}^{\infty} a_{n,r}\right]$$

$$\frac{1}{m}\sum_{r=2}^{\infty}\left[\sum_{n=2}^{\infty} n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta) a_{n,r}\right]$$

By (9):

$$\leq \frac{1}{m}\sum_{r=2}^{m} \alpha(1-\Omega)(1-A)$$

$$\leq \alpha(1-\Omega)(1-A)\frac{1}{m} m$$

$$\leq \alpha(1-\Omega)(1-A)$$

This completes the proof.

Theorem (6):

Let $f_r \in RHB(\delta, \alpha, A, \beta, \lambda, \ell, \Omega), (r = 2,3,\dots,m)$, then

$$g(z) = \sum_{r=2}^{\infty} c_r f_r(z) \in RHB(\delta, \alpha, A, \beta, \lambda, \ell, \Omega)$$

For $f_r(z) = z - \sum_{n=2}^{\infty} a_{n,r} z^n$

Where $\sum_{r=2}^{m} c_r = 1$

Proof: $g(z) = \sum_{r=2}^{m} c_r f_r(z)$

$$= \sum_{r=2}^{m} c_r \left[z - \sum_{n=2}^{\infty} a_{n,r} z^n\right]$$

$$= \sum_{r=2}^{m} c_r z - \sum_{n=2}^{\infty}\sum_{r=2}^{m} c_r a_{n,r} z^n$$

$$g(z) = z - \sum_{n=2}^{\infty} \ell_n z^n$$

Where $\ell_n = \sum_{r=2}^{m} c_r a_{n,r}$

Thus

If $g(z) \in RHB(\delta, \alpha, \mathrm{A}, \beta, \lambda, \ell, \Omega)$

$$\sum_{n=2}^{\infty} \frac{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)}{\alpha(1-\Omega)(1-A)} \ell_n \le 1$$

That is, if

$$\sum_{n=2}^{\infty} \sum_{r=2}^{m} \frac{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)}{\alpha(1-\Omega)(1-A)} c_r a_{n,r} \le 1$$

$$\sum_{r=2}^{m} c_r \sum_{n=2}^{\infty} \frac{n \frac{n!\Gamma(\delta)}{\Gamma(n+\delta-1)} (\ell(n-1) - \lambda\beta)}{\alpha(1-\Omega)(1-A)} a_{n,r} \le \sum_{r=2}^{m} c_r$$

This completes the proof.

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