The Emerging Role of Topological Materials in Condensed Matter Physics: A Review

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Abstract: Remarkable changes have occurred in condensed matter physics over the past several years with development of topological materials which offer a new way of thinking about phases of quantum systems in terms of topological invariants and quantum phases of matter beyond the traditional ideas involving the breaking of symmetries. The aim of this review is to give a comprehensive theoretical guide to the foundations of the topological phases including topological insulators semimetals and superconductors. It also describes the mathematical instruments to describe these phases e.g. topological invariants and the theory of the Berry phase and the effective quantum field tools, which are important in determining the stability of these materials to external perturbations. Classification systems including the AltlandZernbauer symmetry classes and the periodic table of topological phases will also be discussed as we attempt to understand the existence underlying relationship between symmetry and topological structure in the generation of unconventional quantum states. We also inform about the most noticeable theoretical frameworks, which lead to this field development such as the Haldane model and the KaneMille model and mention some of their applicable domains in new domains like quantum computing spintronics and energy-efficient technologies. Finally, we put into perspective the main technical issues confronting the field forced to draw such issues as the challenges that surround the preparation of materials and the control of their properties to such challenges as complex electronic interactions and environmental stability.

***Keywords*:** Condensed Matter Physics, Topological Materials, Topological Invariants, Berry Phase, Quantum Field Theory.

1. Introduction

Condensed matter physics has been one of the pillars of modern physics both in terms of theory and experiments. It deals with investigation of physical characteristics of materials in different conditions both solid and liquid state and gaseous state. Within the last couple of decades due to scientific developments topological materials have become a significant area of interest gathering attention of relevant research and industrial groups as a result of its very distinctive qualities which fail to be described by the established paradigms of materials physics [1] [2] . Their topological properties are very deep in nature and includes notable features like how they will retain their own quantum phases even with the existence of external disturbances and this makes them stable on the microscopic level [1] [3].

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This area is viewed as a potential one with regard to its use in the future, i.e. quantum electronics, the design of high performance, energy efficient components [2], [4]. Topological materials have attained a growing level of interest due to the opportunity to transform the way we think about quantum systems, through the introduction of a new conceptual system of knowledge based on topology, which permits the explanation of novel states, including topological insulators, and Dirac and Weyl points [3], [5].

This occurred in the late 2000s with the discovery of the topological insulators and the field became a point of focus [6]. These materials have unusual properties the most attention-catching one is surface or edge electrical conductivity but at the same time the material acts as an insulator internally [6] [7]. This is attributed to the quantum properties, which were identified by carrying out a close examination of compounds that do not have time-reversal symmetry. This finding amounted to an important qualitative change in appreciating solid-state physics [7] [8]. Since that time the field of study has been extended to new topological classes such as conductors and topological semimetals. The variety of study on quantum matter has contributed to the advancement of more sophisticated theoretical frameworks to understand the state of quantum matter [2] [3] [9].

The purpose of the current paper is to give an in-depth overview of the theoretical basis of the physics of topological materials in terms of the most significant fundamental mathematical tools contributing to the explanation of its complicated phenomena [10].

It will deal with the criteria by which such materials are classified by their various symmetry properties as well as how the properties of such materials can be discovered by the topological number theory [5] [11]. Besides this paper covers the possible use of these materials in other applications like quantum electronics and stable superconductivity as they possess some unconventional physical properties [4] [9].

The research issues impeding the application of these materials in advanced technology namely, quantum computing and low-power electronics will also be discussed [9] [12].

In this work we will seek to elucidate on the potential of topological materials in the emergence of the sphere of condensed matter physics and offer a theory that will facilitate scientific knowledge enhancement in the field of topological materials as well as on future research work in the expanding field [1] [10].

2. Theoretical foundations of topological materials

The topological materials within the theoretical knowledge level is a fundamental framework and a significant aspect when it comes to the advancements in condensed matter physics. Topology in this context, tries to investigate the quantized nature of electronic wave functions and the general properties which are invariant to the effects of infinitesimal variations to the parameters of a system. This differs with the traditional notions of local order. These perfections can be explained by topological invariants like Chern number and the Z2 index which reveal the quantum mechanical properties of these systems in intrinsic manner. The topological materials possess the quantum phases that are robust to the local perturbations such as structural defects, scattering and interference with the outside realm. That stability is caused by the topological protection of some fundamental symmetries such as time-reversal symmetry and crystalline symmetries. Acquiring this feature they brought about a growing interest on these materials in developing fault-tolerant platforms of the quantum computing technology as well as in low-power electronics applications [13]

Due to the significant advance in theoretical frameworks instrumental by the use of machine learning and high-performance computing methods it is evident that the landscape of classifying topological phases has increased immensely no longer being restricted only to topological insulators but being extended further to topological semimetals superconductors and higher-order topological phases [14].

2-1 Topological Insulators

Topological insulators (TIs) are a class of quantum materials that are insulating in the bulk (the interior) but possess conductive states at the edges or surfaces. These edge states are protected by time-reversal symmetry (TRS) giving them a pronounced resistance to non-magnetic disturbances resulting in a spin-polarized non-dissipative electronic transition at the boundaries. The presence of these conductive states can be expressed through a precise mathematical description based on topological constants most notably the Z2 index in two- and three-dimensional systems that resist breaking time-reversal symmetry. The general form of this index can be written in two dimensions as follows:

$$\boldsymbol{v} = \frac{\mathbf{1}}{\mathbf{2\pi}}\left(\int_{EBZ} \boldsymbol{d^2\,K\,\Omega_z} - \oint_{\partial EBZ} \boldsymbol{dk\,.\,A}\right) \boldsymbol{mod}\ \mathbf{2}. \quad (\mathbf{2-1})$$

Where:

v: represents a topological number

Ω: (Berry curvature)

A: (Berry connection)

EBZ: (Effective Brillouin Zone)

∂EBZ: the boundary of the Effective Brillouin Zone

A value of 1 labels a nontrivial topology phase and a value of 0 a conventional insulator. This relation gives the relationship between the topology and the geometrical properties of the wave function through Berry phase. Over the past few years great design and discovery efforts of new topological insulators have been achieved due to the developments of symmetry analysis tools spin-orbit coupling control and first-principles calculations. Such success has been specifically real on layered van der Waals materials and artificial hetero structures. [15] [16] . Such advancements have opened up the possibility to integrate these materials in to many applications such as spintronic devices quantum anomalous Hall platforms and topological qubits in quantum computing [17].

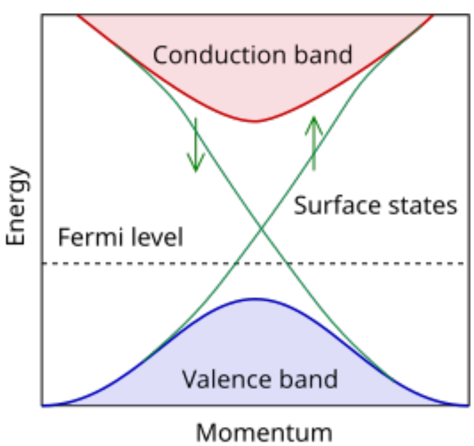

Fig.(1): Band structure of a 3D symmetric topological insulator[17].

2-2 Topological Semimetals

The sample application of topological conductors can be discussed as the example of Dirac and Weyl semimetals that present specific quantum properties. These materials are expressed in the crisscrossing of the conduction and valence bands at discrete positions in the Brillouin zone. These details produce quasiparticles which behave as relativistic Dirac or Weyl fermions giving rise to novel phenomena including surface Fermi arcs and chiral anomalies in the presence of an applied magnetic field[2] [5] [9].

The systems enjoy unusual responses to external perturbations and such aspect has been substantiated by the abundance of theoretical models and experiments [1] [2] [14].

One can describe the energy dispersion of quasiparticles in momentum space as follows:

$$\boldsymbol{E(k) = \pm K|\hbar v_f|} \qquad \boldsymbol{(2-2)}$$

Where:

E(k): energy of the particles versus k=1/wavelength.

h: Planck constant divided by 2pi.

v_f: The Fermi velocity v_F which is the velocity of Weyl-like or Dirac-like particles in the material.

The type of spectrum is ±, which means that it consists of two symmetric bands a positive energy band and a negative energy band at a meeting point which is called the Dirac point or the Weyl point.

2-3 Topological Number Theory (Topological Invariants)

Topological constants are very useful and have a center-stage of characterizing topological phases by their inherent structure in quantum properties. Such constants as Z2 index on topological insulators and Chern number on topological conductors are employed as quantitative measurements that cannot be affected by continuous distortions on the system such as crystalline defects on the lattice and certain phase transitions of the system. These constants describe the robustness of topological phase and its extension in parameter space rendering them highly useful as a way of characterising the structure of non-conventional quantum phases [6] [11] [14] .

F(k) is the Berry curvature within the Brillouin zone and one can calculate Chern by doing an integral thereof as follows:

$$\boldsymbol{C = \frac{1}{2\pi}\int_{BZ} f(K) d^2K} \quad \boldsymbol{(2-3)}$$

with F(k) the Berry curvature in the wave space.

Such numbers cannot be disturbed by the common disturbances, like crystal imperfections or phase changes. This means that this material will retain its topology characteristics despite the regional modifications e.g. influence. Such numbers can describe the presence of inert. edge states in topological insulators.

2-4 The Role of Band Theory

Band theory Solid-state physics Band theory forms a foundation of solid-state physics by offering an explanatory conceptual framework of the electronic configuration of crystalline materials. The band theory is founded on the time-stable solutions to Schrodinger equation. The electrons are supposed to move in a periodic potential produced by the periodic packing of atoms in a crystal lattice. The origin of this periodic potential is the translational symmetry of crystals which permits quantizing the energy levels into distinct bands:

$$\boldsymbol{\hat{H}\psi_{n\,k} = E_n(k)\,\psi_{n\,k}} \quad \boldsymbol{(2-4)}$$

$\boldsymbol{\hat{H}}$ represents the time periodic Hamiltonian $\boldsymbol{\psi_{n\,k}}$ The Bloch wave function in the n n th energy band, the crystal momentum $\boldsymbol{K}$ and the energy dispersion relation $\boldsymbol{E_n(k)}$

The band theory is also useful in the topological material study since this allows us to chart non-trivial band inversions that commonly arise out of spin-orbit coupling or breaking of the crystals. The net outcome of such inversions are leads to surface or edge states that are immune to conventional materials [9] [13].

Highly robust states Spectra of topologically-protected states tend to be very robust to perturbations. This activity is pertinent to the topological invariants on the basis of universal aspects of Bloch wave functions within the Brillouin zone. The recent advances have allowed to build on the achievements of classical band theory, by embodying the symmetry indices in the notions of topological quantum chemistry. The systematic understanding that this integration offers to characterization of topological phases in terms of the eigenvalues of symmetry elements and the content of atomic orbitals at high-symmetry points in the inverse space [14] [18] [20].

These recent methods have allowed both massive and highly effective topological surgery of topological phases, and the building of comprehensive materials databases, which is done utilizing first principles calculations and symmetry-based queries [1] [16] [19] [21].

These more fancy constructions have also led to the discovery of novel topological insulators semimetals and superconductors [20] [21].

The techniques have equally been important in exploring hetero structures especially two-dimensional van der Waals materials that have proven to exhibit tunable and controllable topological properties [15] [17].

2-5 Berry Phase and Topological Curvature

The Berry phase is a geometric quantity obtained by the wave function of a quantum system as it slowly (adiabatically) evolves along a closed path C in parameter space. This phase is defined by the relationship:

$$\boldsymbol{\gamma_n(C) = i\oint_C dR.\langle\psi_n(R)|\nabla_R|\psi_n(R)\rangle} \quad \mathbf{(2-5)}$$

Where:

R represents the vector of coefficients.

∇_R a gradient operator in the parameter space.

$\langle\psi_n(R)|\nabla_R|\psi_n(R)\rangle$ Perry's Connection.

The Berry curvature can be analogized to the behavior of a magnetic field in momentum or parameter space and is a fundamental factor in the modern theory of electric polarization and many topological phenomena. For example it is essential for understanding the anomalous Hall effect in systems that do not possess time-reversal symmetry. It also contributes to the emergence of topologically protected edge and surface states by generating quantized topological constants such as the Chern number [6] [9].

Recent research has revealed a broad role for the Berry curvature beyond classical settings further enhancing its status as a fundamental analytical tool in condensed matter physics:

- In antiferromagnetic systems multipole Berry bend structures have been identified for example in Kagome antiferromagnets such as FeSn. It has been confirmed that the Berry bend quadrupole induces a large third-order nonlinear Hall effect even at room temperature. [22].

- The Berry curvature in non-Hermitian photonic systems has been measured directly using polarimetry in photonic crystal sheets. This allows the valley Chern number to be extracted without affecting the eigenstates of the system. This approach provides a rich experimental understanding of the geometric properties of dissipative or excited systems. [23].
- Moreover some recent studies have shown how the quantum geometric tensor combining Berry curvature and quantum gradient controls nonlinear transport states. It has been shown that systems lacking inversion symmetry exhibit Berry curvature dipoles and quadrupoles leading to second- and third-order responses such as the anomalous nonlinear Hall effect.. [24].

All these results highlight a fundamental shift in our understanding of Berry curvature. Once considered primarily in the context of linear responses and isolation phases it has now become a fundamental element in nonlinear transport photonic topological platforms and quantum engineering.

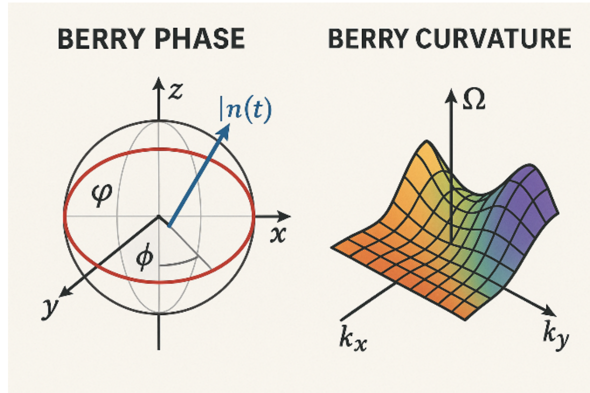

Fig. (2): Representing the concepts of Berry phase and Berry curvature.[Figure designed by the author]

3. Mathematical Techniques Used in the Study of Topological Materials

Understanding topological phases of matter requires a precise mathematical framework capable of describing quantum constants that remain invariant under continuous deformations. This section outlines the main mathematical techniques used to classify and characterize topological materials

3.1 Topological Number Theory (Topological Invariants)

Topological constants are quantized quantities that serve as identifiers for different topological phases. The most prominent of these constants are the Chern number the Z2 indices and the spin numbers. These constants remain stable relative to the smooth deformations which leave the structural symmetry and gap in energy of the system unchanged. Z2 indices Z2 indices are a crucial mechanism which is applied in the classification of topological insulators which have two or three dimensions and preserve time-reversal symmetry. This constant can be typically built up using integrals over the Berry continuum or the Berry twist on a manifold in momentum space or parameter space. This integral can be simplified as given below:

$$\boldsymbol{Z_2 = (-1)\int_M A_d = \pm 1} \qquad \boldsymbol{(3-1-a)}$$

Where:

M represents the manifold in the phase space to the matter

A_d is the differential form coming along with the quantum field connection (Berry connection or Berry curvature in some contexts)

The value of Z2 index is negative 1, which means unconventional topological phase with the occurrence of surface or edge protected states amidst total energy gap. It is recently that it has been extended to the application of quasi-quantum Hall systems which goes to show its high accuracy even in metallic and semi-metallic systems [25]. Among the topologically most significant quantities in systems without the time-reversal symmetry as the quantum Hall effect, Chern number is one of them. This figure is given by the integration of the Berry curvature as expressed by the statement in Equation (2-3). A non-zero Chern number indicates the presence of quantum Hall conductivity and topologically protected chiral edge states. Some recent research has demonstrated the existence of anomalous quantum Hall (QAH) phases with high Chern number in specially designed materials such as $MnBi_2Te_4$ thin films [26]. In one-dimensional systems with chiral or particle-hole symmetry such as the Su-Schrieffer-Heeger (SSH) model the spin number ν provides a criterion for identifying the topological phase

$$\boldsymbol{v = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{d}{dk} \log det[q(K)]\, dk} \quad \boldsymbol{(3-1-b)}$$

Here q(k) is the off-diagonal mass of the Bloch function in its chiral symmetric form. This constant correctly predicts the emergence of zero energy edge states in topologically non-trivial phases. [27].

3.1.1 Emerging Techniques and Generalizations

- **Local Chern Markers**

To link theory and experiment together local Chern signals were developed as an analogue of the Chern number in real space. These signals allow precise spatial detection of topological order which is very important in inhomogeneous or disordered systems. [28].

- **Topology in Non-Hermitian Systems**

Recent theoretical developments have led to an expansion of the concept of topological invariants to non-Hermitian systems where the Hamiltonian is not equal to its conjugate. In these frameworks spectral degeneracies known as exceptional points play a role similar to Dirac nodes in Hermitian systems. The topological invariants in these systems have been redefined in terms of generalized spin numbers and biorthogonal Berry phases which fit the non-standard mathematical properties of the non-Hermitian energy spectrum.. [29].

- **Curvature Renormalization Group (CRG)**

The CRG approach describes the behavior of Berry curvature gradient around phase transitions. The method is an alternative to the traditional renormalization techniques, and it has been demonstrated to be useful in a classification of critical points in a topological system. [30].

3.2 Effective Quantum Field Theory

Effective Quantum Field Theory (EQFT) is the important resource in the area of theoretical physics in comprehending the physical phenomena at lower energies than in the higher energy scales like grand unification energy or the Planck energy. This theory is premised on the gauge separation principle which presupposes that high-energy degrees of freedom can be ignored and be substituted by effective terms that are added to the low-energy Lagrangian. As an illustration in the electroweak energy range a diagrammatic (e.g.: the HooftVeltman diagram technique) can be applied so that renormalization can be studied at the single-loop order and chiral symmetry maintained leading to a consistency of physical amplitude calculation throughout the effective frame [31].

In the subject of condensed matter physics the dynamics of electrons on or close to the Fermi surface can be described in terms of an effective theory grounded upon low-momentum scattering which allows electronic phenomena in crystals to be effectively discussed:

$$L_{eff} = \psi^{+}\left(i\partial_t + \frac{\nabla^2}{2m^*}\right)\psi + \frac{g}{2}(\psi^{+}\psi)^2 + \cdots (3-2)$$

Where:

m^* is the effective mass of the electron

g represents the strength of the interaction between the electrons.

This type of description is classified as a "Fermi-Landau liquid model " an effective field theory for electrons in metals.

3.3 Edge State Theory

Edge state theory describes the behavior of electrons within the boundaries of topological materials such as topological insulators or systems that exhibit the quantum Hall effect. In these cases conductive edge states emerge that resist scattering. This phenomenon is due to the topological structure of the material or its fundamental symmetries. These states arise as a result of the presence of an energy gap in the electronic structure of the crystal with lower-energy energy states remaining localized at the edges. The Bernevig–Hughes–Zhang model and the two-dimensional Schrödinger and Dirac models are the fundamental theoretical models for studying these phenomena. The motion of electrons at the edges can be described using the one-dimensional Dirac equation which describes the propagation of topological conductive states along the boundaries of the sample:

$$\boldsymbol{H = -i\,\hbar v_F \sigma_Z \frac{\partial}{\partial x}} \quad (\boldsymbol{3-3-a})$$

where:

H: the Hamiltonian

$\hbar$: the reduced Planck constant

vF : the Fermi velocity

σ_Z : the Pauli operator

$\frac{\partial}{\partial x}$: the spatial derivative along the edge direction.

This equation expresses the case of a spectral edge moving in a specific direction without the possibility of back reflection. This is due to the separation of the right and left channels due to the topological structure. In the case of the quantum Hall effect the edge can be defined as a channel that conducts current in only one direction. The quantum conductivity can be given by the following equation:

$$\boldsymbol{\sigma_{xy} = v\frac{e^2}{h}} \qquad (\boldsymbol{3-3-b})$$

Where:

σ_{xy} : the Hall conductance

ν : a whole number (in classical quantum effects) or fractional number (in fractional quantum effects) representing the number of edge states.

e : the charge of the electron

h : Planck's constant.

The bulk-boundary correspondence principle states that the number of edge states in a topological system reflects its fundamental topological number such as the Chern number. This principle is the conceptual link between the microscopic topological structure of the system and the observable physical phenomena at the edges. Recent experimental measurements have confirmed this principle particularly in magnetic topological insulators such as $MnBi_2Te_4$ where strong edge currents have been observed even in the absence of external magnetic fields confirming the intrinsic nature of topology in determining edge properties [32].

3.4 Degenerate Matrix Theory

Degenerate matrix theory is a fundamental mathematical tool in the analysis of quantum systems particularly in the study of topological materials that exhibit properties resulting from symmetry and symmetry breaking. A degenerate matrix is a matrix with degenerate (repeated) eigenvalues a property that indicates the presence of internal symmetries in the physical system under study. This degeneracy contributes to the repetition of energy levels a behavior that can be observed in many topological models such as topological insulators and non-classical quantum phases. When the Hamiltonian matrix H is degenerate it contains at least two identical eigenvalues indicating an underlying symmetry in the system that affects its spectral structure.

$$\boldsymbol{H\psi_i = E\psi_i \quad H\psi_j = E\psi_j \ \ where\ i \neq j} \quad (\boldsymbol{3-4-a})$$

ψi and ψj denote wavefunctions with the same eigenvalue E demonstrating spectral degeneracy. It is a direct consequence of quantum degeneracy that also is central to define the topological phase transitions. In order to display the consequences of a perturbation which might result in partial breaking of the symmetry the projection is calculated on to the subspace of degenerate eigenvalues of that is that the eigenvalue E. This extrapolation is written mathematically as the projection matrix P which projects the dynamics onto this subspace and calculates the sensitivity to perturbations in each dimension of this subspace:

$$\boldsymbol{P = \sum_{n=1}^{d} |\psi_n\rangle\langle\psi_n|} \quad (\boldsymbol{3-4-b})$$

Where: ψ_n is a collection of gradient wave functions and d is the number of such functions.

Through the examination of the small weak couplings degenerate perturbation theory is applied to examine small shifts of the energy level and time development of quantum states. This method permits the isolations of hidden dynamical dynamics in degenerate spaces at which the eigenvalues are identical without a perturbation and accessing how the eigenvalues are split, or alteration of the wave functions under small external forces:

$$\boldsymbol{H^{-} = H_0 + \lambda V} \qquad (\boldsymbol{3-4-c})$$

Where: H_0 is the gradient Hamiltonian and V a small perturbation.

Energy transformation can be determined during solving of the interference matrix within the gradient space:

$$\boldsymbol{W_{mn} = \langle\psi_m|V|\psi_n\rangle} \qquad (\boldsymbol{3-4-d})$$

The most recent work demonstrated that this can be done effectively by applying degenerate perturbation theory with the tools of quantum engineering to characterizing both topologically degenerate states and symmetry-protected states [33].

Further studies on exceptional point physics have demonstrated that Hamiltonians that are degenerate behave nonlinearly where their spectral sensitivities attain $\sqrt{L\ \epsilon}$This has the potential of opening up to exciting applications in ultra-precision sensing in non-Hermitian topological systems [34].

The approach is common in the study of quantum phenomena in materials whose topological properties can be highly non-trivial such as systems with strong interactions and symmetry-guaranteed edge states.

3.5 Fourier Transformations and Phase Transitions

Fourier transforms play a fundamental role in the analysis and study of physical systems amenable to spectroscopic description particularly in the context of phase transitions. They allow the transition from real space (r) to reciprocal space (k) facilitating the study of microscopic structures and the behavior of matter in different phases. The three-dimensional Fourier transform of a physical field function f(r) can be written as:

$$\boldsymbol{f^{\sim}(K) = \int_{R^3} f(r)\, e^{-ik.r} d^3r \quad (3-5-a)}$$

While the inverse transformation is given by:

$$\boldsymbol{f(r) = \frac{1}{(2\pi)^3}\int_{R^3} f^{\sim}(K)\, e^{-ik.r} d^3r (3-5-b)}$$

Within the scope of phase transitions Fourier transforms are an essential tool for interpreting correlation functions that describe physical variables interacting (e.g. density or magnetism) in distinct points in a system. The two point correlation function follows as:

$$\boldsymbol{G(r) = \langle \emptyset(0)\emptyset(r)\rangle \quad (3-5-c)}$$

where $\emptyset(r)$ is a physical quantity, magnetism or density and $\langle\cdot\rangle$ refers to ensemble average

Fourier transform is given by:

$$\boldsymbol{S(K) = \int G(r)\ e^{iK.r} d^3r \quad (3-5-d)}$$

The structure factor is measurable through experiment such as X-ray and neutron diffraction. Most recently inverted structure factor calculation under light excitation e.g. in VO 2 has been shown to be possible with tensor network simulations that can furnish good understanding of phase transitions in addition to the spatial correlations in strongly coupled system [35].

4. Classification of topological phases

Topological phases are an unorthodox phase of matter system. They cannot be categorized according to the local properties like energy gap or symmetry breaking unlike the conventional phases. Instead they are characterized by properties that are global in nature resulting out of the way the electron wave function is characterized by the frequencies of the band. These phases themselves are described in terms of mathematical abstractions based on topology and the theory of group representations and are characterized by topological invariants such that these do not change with continuous deformations of the system provided the spectral gap does not cease.

Such a distinction depends both on topology of the reciprocal space, and also on the correspondence between the electronic energy bands. The topological constants are read off and obtained by integrations over the Brillouin zone.

The most prominent example of that is a Chern number that can be estimated in Equation (2-3) and can be viewed as a quantitative measure of the topological nature of the system [13] [19] [36].

4.1 Main types of topological phases:

4.1.1 Topological Insulators

These materials have a spectral gap in their overall electronic structure and at the same time contain conductive edge states that are generated as a direct result of time-reversal symmetry (TRS) which gives them topological protection against non-magnetic disturbances. These materials are classified according to their topological Z_2 indices in two and three dimensions which has recently been proven thanks to the rapid progress in the study of intrinsic topological insulators. [6] [9] [17] [25] [39].

4.1.2 Topological superconductors

Unconventional edge states are found in topological superconductors the most apparent ones are zero-order Majorana modes emerging at system boundaries or topological defects. These systems are categorized by topological indexes of type either Z or Z2 depending on the symmetries the system preserves e.g. time-reversal symmetry and particle gap symmetry. This type of materials was recently a subject of great attention as it is capable of bearing strong anti-disturbance quantum states and therefore will be a future candidate to use topological quantum computing [7] [10] [11] [27] [36] [37].

4.1.3 Topological semimetals

Weyl and Dirac semimetals are among the most important topological materials as they exhibit distinctive spectral points called Weyl or Dirac points near the Fermi level. These points are associated with topological charges such as non-zero Chern number which causes the appearance of Fermi arcs on the crystal surfaces. This spectral structure generates distinct electron transport nonlinear Hall effect and the quantum distortion effect. Angular resolved photon emission spectroscopy (ARPES) and quantum fluctuation data including the de Haas van Alpen effect has been used experimentally to determine the existence of these surface states[2] [5] [22] [38] [39].

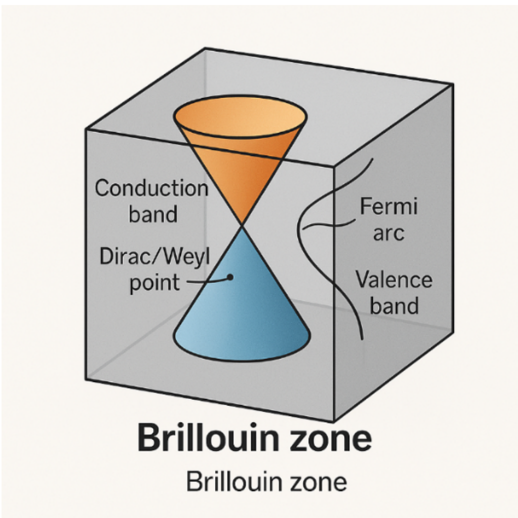

Fig.(3): Energy bands Dirac/Weyl points and Fermi arcs in topological semimetals within the Brillouin Zone [38].

4.1.4 Topological Crystalline Insulators

The topological phases differ in that they are defined in terms of explicit crystal symmetries (reflection, rotation and chirality) rather than being restricted to time-reversal. These states have been experimentally verified in materials including SnTe which is viewed as a sophisticated sample of topological-insulators secured in point reflection invariance. The recent studies have helped in enhancing the topological classification of these phases with the assistance of the crystal symmetry indices and breaking down band representations on high-symmetry points in the Brillouin zone.[14] [18] [21] [40].

4.2 Topological classification and symmetry: the Altland-Zirnbauer family

AltlandZirnbauer (AZ) classification, AZ classification The AZ classification is an important theoretical tool to describe quantum topological phases in terms of three broad classes of symmetry:

- time-reversal symmetry (T)
- Symmetry on particles spacing (C)
- Charge/conductivity symmetry (S)

This typology returns ten topological classes which are all referred to as the periodical table of topological phases. Every spatial dimension has so-called topological index either integer ($\mathbb{Z}$) or binary ($\mathbb{Z}_2$) [10] [11].

				δ							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 1: Periodic table of topological insulators and superconductors (1D up to 3D) [42]

Over the past few years this framework has been extended considerably to incorporate crystalline symmetries higher-order boundary phenomena and non-Hermitian systems leading to the development of extended versions of the AZ table which are currently in use to classify higher-order topological insulators and superconductors [41].

5. Comparison with traditional materials

Topological materials represent a paradigm shift in the classification of condensed matter phases transcending the traditional framework that divides materials into conductors semiconductors and insulators based on local electronic properties particularly the presence of an energy gap at the Fermi level. Topological materials are unique in that they possess universal topological constants such as the Chern number and the Δ_2 index which remain quantized and stable under the influence of simple system distortions as long as the spectral gap does not close [11] [25].

This topological protection provides a significant degree of robustness to structural disturbances and defects unlike conventional materials which rely on symmetry breaking and local order parameters to determine their physical properties. One of the most distinctive features of these materials is the presence of topologically protected surface or edge states which are generated as a result of the interaction of bulk with boundaries in the system. These states are established by time-reversal symmetry or specific crystal symmetries and remain stable even in the presence of moderate disturbances [9] [14].

In contrast surface states in conventional materials exhibit high sensitivity to impurities and defects and their conductive properties can easily degrade. Furthermore topological materials exhibit exceptional quantum phenomena such as dissipative edge transport and quantum conduction which are not observed in conventional systems based on quasi-classical charge carrier motion. These properties provide a gateway to advanced applications in fields such as fault-tolerant quantum computing spintronics and highly energy-efficient nanoelectronics [9] [17] [36].

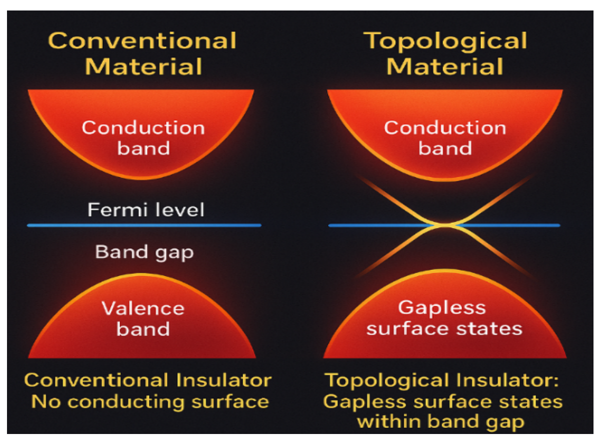

Fig. (4): Emergence of protected surface states in topological materials compared to conventional materials[.[Figure designed by the author]

6. Modern Applications and Theoretical Models

Topological materials have emerged as a key element in contemporary condensed matter physics due to their protected surface states and unique symmetry properties which open the way for pioneering applications in theory and technology.

6.1 Prominent Practical Applications

- **Topological Quantum Computing:** Topological superconductors are one of the most distinctive quantum systems that support zero-point Majorana modes. These quasiparticles exhibit non-Abelian statistics making them fundamentally different from conventional fermions and bosons. These modes provide a promising platform for creating quantum bits (qubits) that exhibit natural fault tolerance due to their topological protection making them ideal for topological quantum computing applications. Recent studies have demonstrated the emergence of these modes in several systems including Mn_2B_2 as well as in carefully engineered hybrid structures that combine superconductivity and electronic topology which will support the possibility of controlling these states and using them in future quantum devices. [36] [37].
- **Low-Power Electronics:** Topologically protected edge states in two-dimensional topological insulators provide a scattering-free electron transport channel due to their resistance to scattering from defects or impurities enabling efficient energy transfer at the nanoscale. This behavior is ideal for developing low-power electronic devices as power loss can be minimized. This makes these materials a promising option for designing energy-efficient nanoelectronic components. [9] [17].
- **Topological Spintronic:** The strong coupling between electron momentum and spin called spin-locking in topological insulators offers promising possibilities for generating stable spin currents without the need for external magnetic fields. This physical principle is a fundamental step toward developing high-performance logic devices and memory characterized by ultrafast speed and stability making topological insulators a promising foundation for modern spintronics.[17].
- **Quantum and Magnetic Sensors:** Weyl and Dirac semimetals exhibit unconventional and strong responses to electromagnetic fields due to the chiral anomaly and Berry curvature effects characteristic of their topological structure. These properties enhance

magnetic and gravitational sensitivity paving the way for precise applications in topological sensing and high-performance detection in complex quantum environments [24] [38] [39].

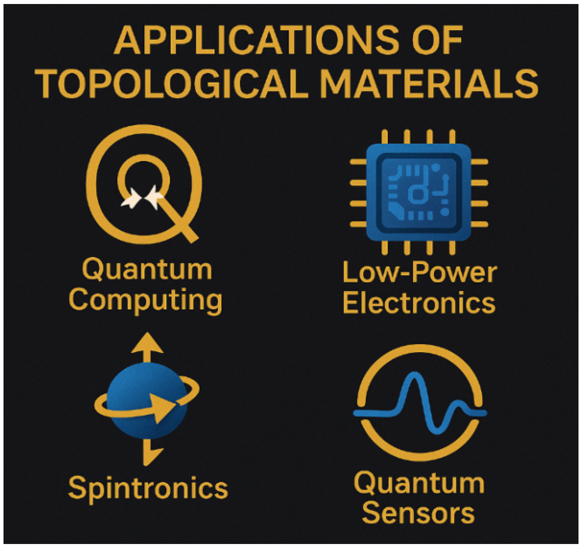

Fig. (5): Main areas of applications of topological materials.[Figure designed by the author]

6.2 Advanced Theoretical Models

The theoretical exploration of topological phases has been developed using sophisticated models that provide deep mathematical insights into the non-trivial topology of domains:

- **Haldane Model (1988):**
  Haldane's (1988) model is one of the fundamental models that explains the emergence of Chern insulators without the need for a net magnetic field which provides a theoretical framework to accommodate the anomalous quantum Hall effect. In Ak (2005) Kane-Milley's model was introduced as an extension of this model by linking the spin-orbit coupling within a similar lattice structure which opened the door to understanding two-dimensional topological insulators especially in materials such as graphene. This model represented a basis for explaining the existence of edge states protected by time-reversal symmetry and contributed to the development of the concept of the topological Z_2 index...
- **Kane-Mele Model (2005):** The incorporation of spin-orbit coupling into the Haldane model opened the door to the discovery of the quantum spin Hall effect in two-dimensional systems similar to graphene. This modification demonstrated the possibility of generating anti-dispersion spin edge currents protected by time-reversal symmetry without the application of an external magnetic field. This discovery established a fundamental step in the development of the theory of topological insulators and their future applications in spintronics and quantum computing. [11].
- **Non-Hermitian Topological Models:** These models expand the scope of the traditional topological classification to include non-Hermitian systems characterized by gain and loss mechanisms. This has resulted in the discovery of new topological phases associated with unconventional edge

states not found in Hermitian systems. These developments draw attention to distinctive spectral structures such as the asymmetric spectrum and exceptional points increasing the understanding of topology in open systems and offering promising possibilities for the design of highly sensitive photonic and electronic devices [23] [29] .

7. Challenges and Future Prospects

Despite the quantum leap in the understanding and design of topological materials fundamental challenges remain that limit their practical applications. The most important of these challenges is the difficulty of manufacturing topological materials with high crystalline quality that can maintain their topological properties under harsh environmental conditions. Factors such as structural heterogeneity temperature fluctuations and pressure can weaken protected edge states and quantum coherence causing a deterioration in topological performance [14] [21] [40].

There is also a significant gap between ideal theoretical models and experimental applications. Many topological predictions rely on simplified proposals for band structures without taking into account electron interactions chaos and lattice vibrations. To bridge this gap advanced approaches such as dynamical mean field theory (DMFT) density matrix renormalization group (DMRG) and quantum Monte Carlo simulations must be employed which provide a more accurate description of systems with strong interactions [31] .

Nevertheless the future of topological materials remains promising. For example topological quantum bits based on Majorana patterns are a prime candidate for building error-tolerant and scalable quantum computers [36], [37]. The potential to reduce energy loss and increase electrical conductivity at room temperature expands the horizon for developing low-power yet highly efficient electronic technologies [9] [22].

These insights can also be advanced with modern techniques in artificial intelligence and high-performance computing which are being leveraged to accelerate the discovery and design of topological materials. Machine learning algorithms are being employed to predict topological invariants from initial crystal structures helping to guide experimental efforts more efficiently [16], [19]. This integration of theoretical physics and data science is likely to yield significant breakthroughs in the near future bringing topological materials closer to practical technical applications.

8. Conclusion

In this work, we detail the theoretical and physical developments associated with topological materials highlighting their fundamental role in redefining fundamental concepts in solid-state physics. Theoretical models such as topological insulators and conductors have created an ingenious conceptual framework for understanding the quantum behavior of electrons in complex systems and have contributed to the discovery of unconventional electronic phenomena based on comprehensive topological properties. We systematically compare topological materials with their conventional counterparts demonstrating the fundamental differences in their electronic magnetic and optical responses as confirmed by recent experimental results. We also highlight emerging trends such as strong electronic interactions symmetry functions and topology in non-Hermitian systems which currently represent active areas of research. The field of topological materials is one of the most dynamic areas in contemporary physics bridging theoretical challenges with promising technical applications. With the continued development of fabrication techniques numerical simulations and the use of artificial intelligence tools these materials can be expected to contribute to the development of advanced electronic and quantum systems. Therefore continuing multidisciplinary research in this field is a necessary step towards exploring the full potential of topological materials and exploiting them in next-generation applications.

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Not applicable.

Informed Consent

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